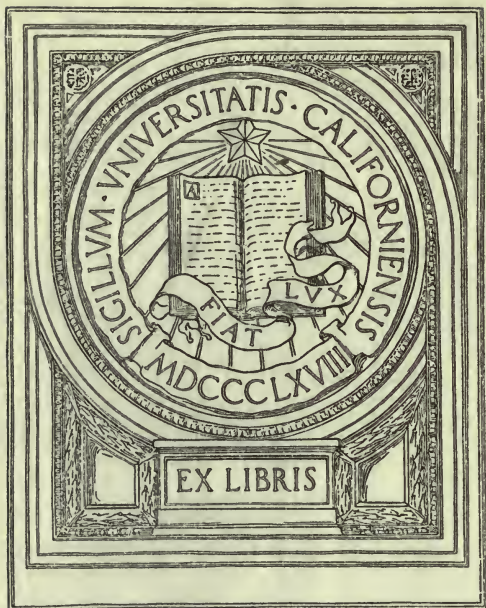
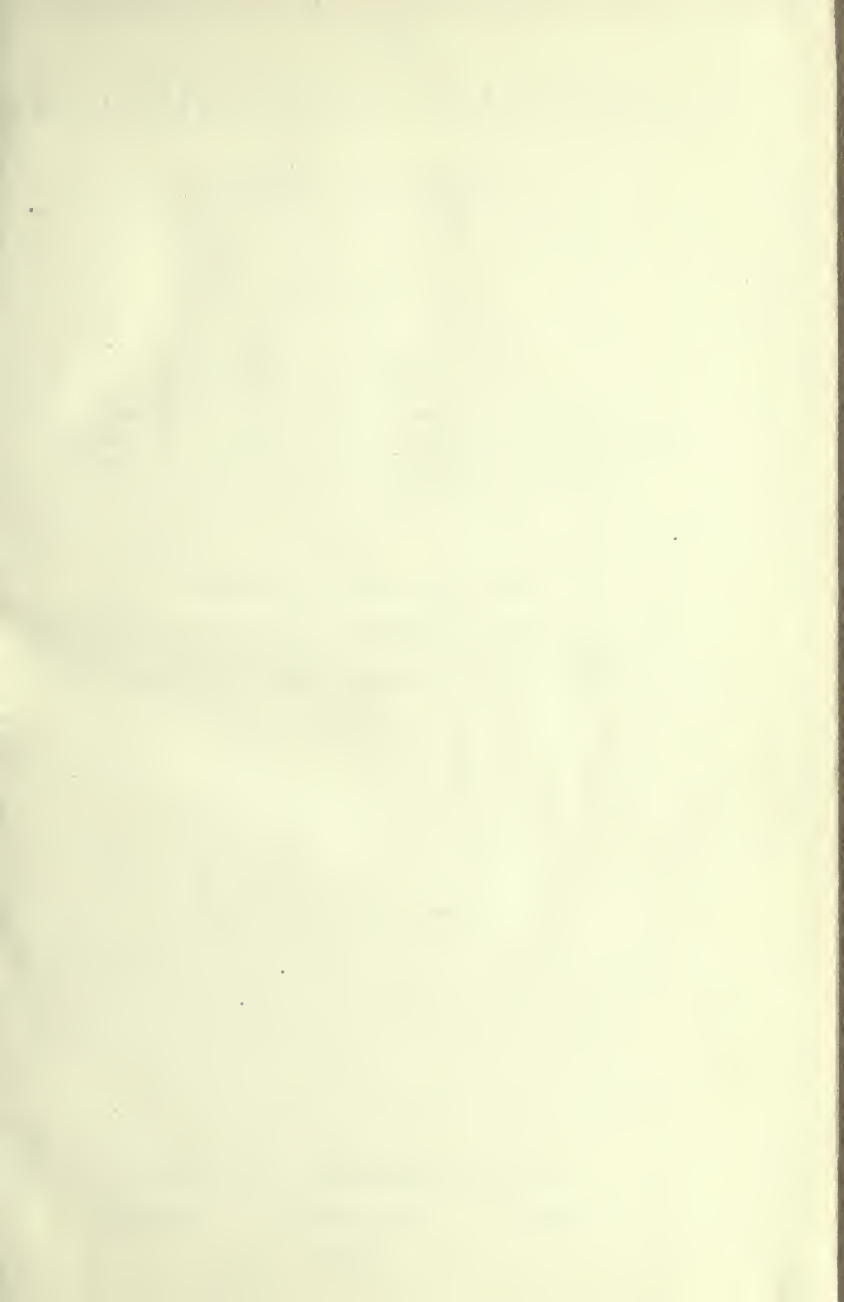








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ELEMENTARY ALGEBRA

FIRST COURSE

BY

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JOHN C. STONE AND JAMES F. MILLIS

PREFACE

THE work in this book is that comprised in the usual first-year course in algebra in secondary schools. The material is so arranged as to adapt the book to the use of those schools that give only a brief course in the subject, as well as schools giving more extended or more difficult courses. A more advanced book, a companion to this, called the *SECOND COURSE*, reviews the important topics covered in the *FIRST COURSE*, amplifies the treatment of these topics where necessary, and covers all additional topics that are required for college entrance.

In the preparation of this book, the authors have combined what teachers have found to be the permanently valuable features of their *Essentials of Algebra* with those features which characterize the most modern ideas in the teaching of secondary school mathematics, and which, through widespread test by application in the classroom, have become generally accepted. It is believed that this text will be found teachable and well balanced. It is free from "fads" and untried experiments, yet sufficiently in harmony with present tendencies to meet the demands of the most progressive teacher.

The book contains a minimum of theory and a maximum of practice. Abundance of drill is provided throughout the book. New processes and principles are psychologically and adequately developed, but they are finally mastered by the pupil through plentiful application in drill exercises and problems.

Throughout the text the work has been planned so as to introduce the pupil to *only one new difficulty at a time*, and this is mastered through copious drill before proceeding to the next. This controlling principle in the development and sequence of subject matter is one of the fundamentally important features

of the book. Observe, for example, the introductory chapter, in which one new point of notation is developed at a time. Observe, also, the entire treatment of the equation.

The pupil is led to see the subject, not merely as a system of exercises to be pursued for the purpose of mental discipline, although the authors are thorough believers in the mental discipline feature of the study, but rather as a scientific instrument for solving certain types of problems such as are actually encountered in the world's work. See §§ 1, 2, 15, 16, 17, 22, etc.

To this end adequate use has been made of real applied problems of the various types encountered in practical life. Thus, in Chapter I computation by use of practical formulæ is introduced. In Chapter II the equation is introduced as a scientific instrument used in the solution of practical problems. In Chapter III many practical applications of negative number are made, etc. These real applied problems lend intense interest to the work. Through them the work is legitimately motivated. Furthermore, it is through the application of the pupil's knowledge of algebra in the solution of these real problems that the knowledge is led to function. The old-time useless puzzles have been eliminated from this book.

Algebra has been correlated with arithmetic and geometry. Review and use of the various process with whole numbers, common and decimal fractions, have been made in the practical computation with formulæ, and in the process of checking work by evaluation of the literal expressions involved. Short methods of multiplication of arithmetical numbers are taught in connection with algebraic multiplication and factoring, as in §§ 75, 77, etc. Principles and processes in arithmetic have been made the starting points in the development of many principles and processes of algebra, as in §§ 50, 51, 54, etc. Considerable use has been made of percentage in the solution of certain types of business problems. The facts of mensuration have been much used, and some empirical knowledge of simple facts in geometry with which the pupil is familiar has been drawn upon in problems.

The graph has been used as a natural means of solving certain

PREFACE

types of problems and of interpretation of algebraic principles, rather than as an excrescence in the form of topics of analytic geometry.

Supplementary exercises have been given at the end of each chapter. They adapt the text to the use of schools with different kinds of courses. These exercises, as a rule, are more difficult than those in the body of the chapters. Schools wanting a brief course may omit these entirely. They may be used for reviews or in classes wanting a more difficult course.

The authors wish to acknowledge their indebtedness to all those whose timely suggestions and criticisms have helped in the preparation of this textbook, and especially to Mr. R. G. Kinkead, Assistant Superintendent of Schools, Columbus, Ohio; who has read critically all of the manuscript.

JOHN C. STONE.

JAMES F. MILLIS.

MAY, 1911.

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ELEMENTARY ALGEBRA

FIRST COURSE

CHAPTER I

THE FORMULA: GENERAL NUMBER

1. **Algebra.**—Of the many kinds of problems encountered in the world's work, some are solved by arithmetic alone, some by *algebra*, and some by other branches of mathematics. **Algebra**, like arithmetic, deals with numbers. There is no clear line of distinction between these two subjects. But in addition to the Hindu symbols of arithmetic, algebra employs letters to represent numbers. In addition to the integers and fractions of arithmetic, it deals with new kinds of numbers. And, as we shall see, it employs some new and interesting principles in the use of numbers.

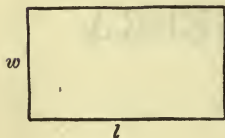
2. **The Formula.**—Many of the *rules* which the student has already encountered in arithmetic, as well as practical rules which are used in the various fields of manufacturing, the trades, the sciences, etc., are expressed often by *formulae*, in which the numbers are represented by *letters*. A formula expresses a rule in a sort of shorthand. This will be seen clearly in the following problems.

EXERCISES

1. How many square feet in a floor 12 feet wide and 15 feet long?
2. A concrete walk is 4 feet 6 inches wide and 24 feet long. Find the area of its surface.

3. What is the area of a rectangle 3.4 inches wide and 7.8 inches long?

4. In general, the rule for finding the area of any rectangle that is l feet long and w feet wide may be expressed by the formula: $\text{Area} = l \times w$ square feet.



Find the area when $l = 3\frac{1}{2}$ feet and $w = 2$ feet. When $l = 16$ feet and $w = 2\frac{1}{2}$ feet. When $l = 50$ feet and $w = 4\frac{1}{2}$ feet.

When $l = 136$ feet and $w = 80$ feet.

3. **Particular and General Numbers.** — In Problem 4, § 2, the letter l is used to represent the length of any rectangle, and hence if we consider all possible rectangles, l may have many particular values. Similarly, if we consider all possible rectangles, w may have many particular values. It is seen, then, that in expressing a rule by a formula, letters are used to represent numbers, and to such letters may be assigned many particular values.

Compared with the **particular numbers** of arithmetic, 1, 2, 3, 4, etc., numbers represented by letters are called **general numbers**.

Numbers represented by letters are also called **literal numbers**.

4. **Factors. Signs of Multiplication.** — The numbers which are multiplied to form a given number are called its **factors**, as l and w in $l \times w$. In addition to use of the sign $\times$ to indicate multiplication, as in Problem 4, § 2, multiplication is sometimes expressed by placing a *dot* between the two factors, halfway up from the lower edge of the number-symbols.

Thus, $l \times w$ may be written also $l \cdot w$.

But unless both of the numbers multiplied are represented by Hindu symbols, the multiplication usually is expressed by *omitting* the multiplication sign.

Thus, the product $l \times w$ is written lw , and $2 \times w$ is written $2w$.

Express in three ways the product of 5 and l .

EXERCISES

1. Express the product of a and b ; of 2, x , and y .

Give the value of $2ab$ when a and b have the following values:

2. $a = 2, b = 3.$ 5. $a = 10, b = 30.$ 8. $a = 25, b = 4.$

3. $a = 5, b = 6.$ 6. $a = 40, b = 20.$ 9. $a = 75, b = 6.$

4. $a = 3, b = 7.$ 7. $a = 50, b = 30.$ 10. $a = 60, b = 3\frac{1}{2}.$

11. Read Exercise 4, page 2, and give a meaning for $A = lw$.

Find the value of A when l and w have the following values:

12. $l = 18, w = 10.$ 14. $l = 13\frac{1}{3}, w = 12.$ 16. $l = 2.4, w = 1.5.$

13. $l = 12.5, w = 8.$ 15. $l = 6.2, w = 4.$ 17. $l = 8.5, w = 3.2.$

18. When b represents the base of a triangle, h the altitude, and A the area, the area is found by the formula $A = \frac{1}{2}bh$. Give the rule which this formula expresses.

In the formula $A = \frac{1}{2}bh$, give the value of A when b and h have the following values:

19. $b = 12, h = 10.$ 21. $b = 20, h = 16.5.$ 23. $b = 16, h = 9.5.$

20. $b = 14, h = 12.$ 22. $b = 8.5, h = 12.$ 24. $b = 12.5, h = 8.$

25. If c is the circumference and d the diameter of a circle, then $c = \pi d$. Compute the circumference of a circle from the formula $c = \pi d$ when $d = 10$. ($\pi = 3.1416$, approximately.)

26. When r = the radius of a circle, $c = 2\pi r$. Find c when $r = 40$; $r = 60$; $r = 7.5$.

27. The volume of a prism is found by multiplying the area of the base by the height or altitude. If V = volume, b = area of base, and h = altitude, then $V = bh$. Find V when

$b = 40$ and $h = 30$; $b = 48$ and $h = 12$; $b = 96$ and $h = 10\frac{1}{2}$.

28. If the base of a prism is a rectangle whose length is l and whose width is w , then $b = lw$. Hence bh becomes lwh . Then we have $V = lwh$. Express the meaning of this formula.

In the formula $V = lwh$, find V when l , w , and h have the values:

29. $l = 42$, $w = 16$, $h = 14$.

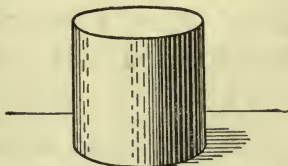
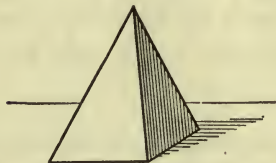
31. $l = 36$, $w = 30$, $h = 18$.

30. $l = 38$, $w = 20$, $h = 12\frac{1}{2}$.

32. $l = 42$, $w = 38$, $h = 12$.

33. The volume of a pyramid is computed by the formula $V = \frac{1}{3}bh$, where V = volume, b = area of base, and h = altitude. State in words the rule which this formula expresses.

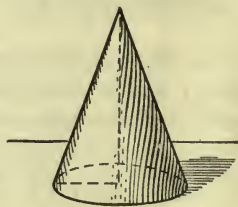
Find V when $b = 12$ and $h = 4\frac{1}{2}$.



34. The volume of a cylinder is equal to the product of the altitude by the area of the base. Express this by a formula.

35. The volume of a cone is one third of the volume of a cylinder of the same base and the same altitude. If V = volume, b = area of base, and h = altitude, write the formula for computing the volume of a cone.

If the base of a cone is 36 square inches, and the altitude 1 foot 6 inches, find its volume.



36. The rule for finding the area of a circle whose radius is r is expressed by the formula, $A = \pi r^2$. Give the rule which this formula expresses.

Find A when $r = 4$. When $r = 10$.

37. Find the area of a circle whose radius is 6 inches.

38. When there is a steam pressure of 90 pounds per square inch in the cylinder of an engine, what is the total pressure on a 12-inch piston? (By a 12-inch piston is meant a piston whose diameter is 12 inches.)

5. Powers. — In Problem 36, § 4, r^2 is a short way of writing rr . Similarly, rrr is written r^3 , $aaaa$ is written a^4 , $10 \times 10 \times 10 \times 10 \times 10$ is written 10^5 , etc. In general, when all of the factors of a product are equal, the product is called a **power** of one of the factors, and is written in this abbreviated form.

x^2 is read “ x square,” or “ x second power”;

x^3 is read " x cube," or " x third power";

x^4 is read “ x fourth power”;

x^5 is read “ x fifth power”; etc.

In a power such as m^7 , the factor m is called the **base**, and the number 7 is called the **exponent**. The exponent tells how many times the base is used as a factor to make the power.

EXERCISES

1. Name the bases and exponents in the following: a^3 , y^{10} , C^{16} , A^{25} , 9^5 , x^n .

2. Express with exponents: $m m m m m$, $b b b b b b b$, $2 \times 2 \times 2$
 $\times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$.

3. Find the value of $2 \times 2 \times 2$; of 2^3 ; of 3^3 ; of 2^4 ; of 3^4 ; of 2^5 .

4. In the formula, $A = \pi r^2$, find A when $r = 6.5$.

5. The area of a circle is found also by use of the formula, $A = \frac{1}{4} \pi d^2$, where d is the diameter. Express this in words as a rule. Find A when $d = 4\frac{1}{2}$.

6. The area of the surface of a sphere is computed by the formula $A = 4\pi r^2$, where A = area, r = radius. Find the area if the radius is 12 inches.

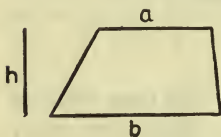
7. The earth is approximately a sphere whose radius is 4000 miles. Find its area.

8. The rule for computing the volume of a cone of which the base is a circle is expressed by $V = \frac{1}{3} \pi r^2 h$, where r = radius of base and h = altitude. Find V when $r = 6$ and $h = 4$.

9. The volume of a sphere is computed by the formula, $V = \frac{4}{3} \pi r^3$, where V = volume and r = radius. Find V when $r = 3$.

10. The diameter of the moon is 2160 miles. Find its volume.

11. The bases or parallel sides of a trapezoid are 6 inches and 10 inches, respectively, and the altitude or distance between them is 8 inches. Find the area.



12. The area of a trapezoid is equal to the product of one half of the altitude by the sum of the bases. This is expressed by the formula, $A = \frac{1}{2} h (a + b)$, where h is the altitude and a and b , respectively, are the bases. Find A when $h = 4$ feet, $a = 3$ feet, and $b = 6$ feet.

6. **Use of Parentheses.**—In the formula in Problem 12, § 5, the symbols $()$, which inclose $a + b$, or the sum of the bases, are called **parentheses**. These symbols are used to inclose two or more numbers which are to be added or subtracted, etc., and indicate that those numbers are not to be used singly, but are first to be combined into one number and then the result used.

Thus, $\frac{1}{2} h (a + b)$ indicates that a and b are first to be added, then the result multiplied by $\frac{1}{2} h$. Similarly, $x(a^2 - b^2)$ indicates that $a^2 - b^2$, as one number, is to be multiplied by x ; that is, we must first square the values of a and b and subtract, then multiply the difference by x . And $(m + n)(m - n)$ indicates that we are first to find the sum of m and n , then their difference, then multiply these results.

In using a formula containing parentheses, *the operations with the numbers within the parentheses must be performed first.*

7. **Other Signs of Grouping.**—There are other symbols having the same meaning as parentheses that are used sometimes to inclose numbers. They are **brackets** $[]$ and **braces** $\{ \}$. A fourth symbol, called the **vinculum**, is sometimes used. It consists of a straight line drawn above the numbers, as $\overline{a + b}$, and is used especially in expressing roots and fractions.

Thus, $B(M - N)$, $B[M - N]$, $B\{M - N\}$, and $B \cdot \overline{M - N}$ all indicate the same thing, viz. that N is to be subtracted from M , and the remainder multiplied by B .

All of these symbols are called **signs of grouping** or **signs of aggregation**.

EXERCISES

1. In the formula $A = \frac{1}{2}h(a + b)$, find A when $h = 4$, $a = 6$, and $b = 7$.

2. Find the value of $d(x - y)$ when $d = 12$, $x = 5$, $y = 1$.

3. Find the value of $(m - n)(m + n)$ when $m = 10$ and $n = 6$.

4. Find the value of $D^2\{D - 5\}$ when $D = 8$.

5. Find the value of $[A + B][A - B][A^2 + B^2]$ when $A = 5$ and $B = 3$.

6. Find the value of $(x + y)^2$ when $x = 4$ and $y = 2$.

7. Find the value of $5(R + S)^2(R - S)^2$ when $R = 10$ and $S = 2$.

8. Find the value of $(g + 10) \div (5 - g)$ when $g = 3$.

9. In the formula, $S = a(t - \frac{1}{2})$, find the value of S when $a = 32$ and $t = 3$.

10. In the formula $A = \pi(R^2 - r^2)$, find the value of A when $R = 14$ and $r = 6$.

11. In the formula $S = 2\pi R(H + R)$, find the value of S when $R = 4$ and $H = 5$.

Letting the hypotenuse of a right triangle $= z$, the base $= x$, and the altitude $= y$, it has been shown in arithmetic that $z = \sqrt{x^2 + y^2}$. Find the value of z when:

12. $x = 3$, $y = 4$. 14. $x = 40$, $y = 30$. 16. $x = 12$, $y = 16$.

13. $x = 6$, $y = 8$. 15. $x = 15$, $y = 20$. 17. $x = 18$, $y = 24$.

In the formula $x = \sqrt{z^2 - y^2}$, find the value of x when:

18. $z = 5$, $y = 4$. 20. $z = 25$, $y = 20$. 22. $z = 50$, $y = 30$.

19. $z = 10$, $y = 6$. 21. $z = 15$, $y = 12$. 23. $z = 35$, $y = 28$.

8. Number Expressions. — Any number symbol, or combination of number symbols, indicating one or more of the operations of addition, subtraction, multiplication, etc., such as those used in the formulæ in the preceding sections, is called a **number expression**, or simply an **expression**.

Thus, πr^2 , $2A - 3B + C$, and $\frac{1}{2}h(b + b')$ are expressions.

An expression involving one or more numbers represented by letters is often called a **literal expression**.

9. Terms. — In the expression $5x^2 - 2xy + 7y^2$, the parts $5x^2$, $2xy$, and $7y^2$ are called **terms**. In general, the parts of an expression connected by the signs $+$ or $-$ are called the *terms* of the expression. An expression such as $6a^2$, or $10x(m - n)$, which is not formed by two or more parts connected by the signs $+$ or $-$, consists of *one term*.

10. Names of Expressions. — How many terms in $5a^2\sqrt{b}$? In $2P^2 - Q^2$? In $4A - B - C$? In $n^3 - 2n^2 + 3n - 1$?

An expression which consists of *only one* term is a **monomial**.

An expression which consists of *two* terms is a **binomial**.

An expression which consists of *three* terms is a **trinomial**.

The name **polynomial** is applied to an expression which consists of *more than one* term.

It must be remembered that an expression within a *sign of grouping* is to be considered as one term.

Thus, $4a - (b - 3c)$ is a *binomial*. Name the two terms.

EXERCISES

Tell how many terms in each of the following expressions, and apply the name "monomial," "binomial," etc., to each:

1. $25A^2BC$.

5. $(m + n)^2$.

8. $110 - 5\frac{H}{D}$.

2. $x^3 - y^3$.

6. $a^4 + b^4$.

9. $\frac{62.5QH}{550}$.

3. $a^2 + 2ab + b^2$.

7. $\frac{1}{6Rt^2}$.

4. $\pi h(r + r')$.

10. $.0622 \frac{W}{V} (v - v')^2.$

12. $\sqrt{A} + C.$

11. $.7854 D^2 p \left(1 - \frac{F}{100}\right) - W.$

13. $y^2 - 5y + 6.$

14. $a^2 - (b - c)^2.$

15. $m^2 + n^2 + p^2 + 2mn + 2mp + 2np.$

16. $R^2 - D^2 + 2DC - C^2.$

18. $r_1 r_2 + r_2 r_3 + r_1 r_3.$

17. $2xy - (x^2 + y^2) + z^2.$

19. $(a + b)(a - b).$

20. $x^3 + 3x^2y + 3xy^2 + y^3.$

11. Evaluation of Polynomials.—The computations by the formulæ in the preceding sections have given practice in finding the values of expressions of *one term*.

In finding the value of an expression containing more than one term, the value of each term must be found before the additions or subtractions indicated between the terms.

Thus, if $x = 2$, $3x^2 - 4x + 7 = 12 - 8 + 7 = 11.$

And, if $a = 12$, $b = 3$, $c = 2$, and $d = 4$,

$$\begin{aligned} 2c(a - b) - (b + c)d &= 2 \times 2 \times 9 - 5 \times 4 \\ &= 36 - 20 \\ &= 16. \end{aligned}$$

Here the values of $a - b$ and $b + c$ are computed first. See § 6.

EXERCISES

1. Find the value of $2P - 3Q + 7R - 2S$ when $P = 10$, $Q = 6$, $R = 1$, $S = 4$.

2. Find the value of $M^2 + 2MN + N^2$ when $M = 4$, $N = 5$.

3. Find the value of $ab + bc - cd - 5$ when $a = 10$, $b = 6$, $c = 1$, $d = 3$.

4. Find the value of $x^3 + 5x^2 - 2x - 3$ when $x = 4$.

5. Find the value of $y^3 - x^3$ when $y = 4$, $x = 3$.

6. Find the value of $(u - v)^2 - (w - x)^2$ when $u = 12$, $v = 8$, $w = 5$, $x = 3$.

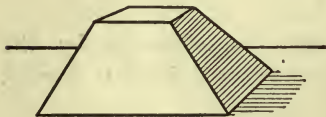
7. Find the value of $[2p + 3q]^2 - [10p - 2q][4p + q]$ when $p = 1, q = 4$.

8. Find the value of $\sqrt{A^2 + B^2 + 2} - \sqrt{C^2 - 75}$ when $A = 5, B = 3, C = 10$.

9. $V = \frac{1}{3} \pi h (r^2 + r'^2 + rr')$. Find V when $r = 12, r' = 15, h = 18$.

10. In § 10, Exercise 8, find value of expression when $H = 96, D = 15$.

11. In § 10, Exercise 11, find value of expression when $D = 18, p = 60, F = 10, W = 1140$.



12. The volume of the frustum of a pyramid is computed by the formula $V = \frac{1}{3} h (b + B + \sqrt{bB})$, where V = volume, h = altitude,

b = area of one base, and B = area of other base. Find V when $h = 6$ inches, $b = 16$ square inches, and $B = 49$ square inches.

In the formula $V = \frac{1}{3} h (b + B + \sqrt{bB})$, find the value of V when:

13. $h = 6, b = 16, B = 25$.

16. $h = 12, b = 49, B = 64$.

14. $h = 8, b = 9, B = 36$.

17. $h = 16, b = 25, B = 36$.

15. $h = 10, b = 25, B = 49$.

18. $h = 18, b = 9, B = 25$.

In the formula $V = \frac{1}{3} \pi h (r^2 + r'^2 + rr')$, find the value of V when:

19. $r = 5, r' = 6, h = 9$.

21. $r = 12, r' = 15, h = 18$.

20. $r = 8, r' = 9, h = 6$.

22. $r = 10, r' = 16, h = 12$.

12. Coefficients.—If an expression is separated into two factors, either factor is called the **coefficient** (co-factor) of the other. If a coefficient is an arithmetical number, it is called a **numerical coefficient**. Otherwise, it is a **literal coefficient**.

In $6xy$, 6 is the numerical coefficient of xy ; x is the coefficient of $6y$; and y the coefficient of $6x$. In $4n(a - b)$, what is the numerical coefficient? What is the coefficient of $4(a - b)$? Of $a - b$?

If no numerical coefficient is written, the numerical coefficient 1 is *understood*.

Thus, since x is the same as $1x$, the numerical coefficient of x is 1. Likewise, the numerical coefficient of PQ is 1, and of $(m - n)$ it is 1.

13. Similar Terms.—In what respect do $2A$, $7A$, and $10A$ differ? Terms which do not differ at all, or which differ only in their coefficients, are called **like** or **similar terms**.

Thus, x^2y , $4x^2y$, and $20x^2y$ are similar. The terms, ax , bx , and cx are similar terms in x .

14. Addition and Subtraction of Similar Terms.—Just as the sum of 2 apples, 5 apples, and 8 apples is 15 apples, so the sum of $2a$, $5a$, and $8a$ is $15a$. And just as $3\text{ men} + 4\text{ men} = 7\text{ men}$, so $3m + 4m = 7m$. Similarly, $6x + 9x = (6 + 9)x$, or $15x$; $C + 5C + 3C = (1 + 5 + 3)C$, or $9C$.

To add similar terms, add their coefficients and to the result attach the common letters with their exponents.

Just as $12\text{ lb.} - 7\text{ lb.} = 5\text{ lb.}$, so $12l - 7l = 5l$. And $32xy - 18xy = (32 - 18)xy$, or $14xy$; $15a^2b^2 - 9a^2b^2 = (15 - 9)a^2b^2$, or $6a^2b^2$.

To subtract similar terms, subtract their coefficients and to the result attach the common letters with their exponents.

EXERCISES

Find the sums in 1-12:

- | | |
|------------------------------|--|
| 1. $2a, 3a, 4a$. | 15. $60z + 25z + 5z + z = ?$ |
| 2. $5x, x, 2x$. | 16. $12d - 4d = ?$ |
| 3. $10W, 3W, 8W$. | 17. $9r_1 - 3r_1 = ?$ |
| 4. $P, 6P, 4P, 2P$. | 18. $16m' - 12m' = ?$ |
| 5. $3t, t, 7t, 6t$. | 19. $13\pi R^2 - 5\pi R^2 = ?$ |
| 6. $5S, 12S, S, 2S$. | 20. $10y - 10y = ?$ |
| 7. $4k, 2k, 9k, 5k$. | 21. $25D - 17D = ?$ |
| 8. $6A, 12A, A, A$. | 22. $69w - 37w = ?$ |
| 9. $3x, 5x, 9x$. | 23. $20 - 8 + 10 = ?$ |
| 10. $mn, 4mn, 3mn, 10mn$. | 24. $12 + 3 - 9 = ?$ |
| 11. $abc, 9abc, 3abc, abc$. | 25. $16 - 2 - 12 = ?$ |
| 12. $4E, 3E, 15E, 10E$. | 26. $6n - 2n + 3n - 4n = ?$ |
| 13. $2a + 6a + 4a = ?$ | 27. $12t + 2t - 7t - t = ?$ |
| 14. $4b + b + 12b = ?$ | 28. $3\pi D + 5\pi D - \pi D - 4\pi D = ?$ |

29. Compare the values of $3+5$ and $5+3$. Of $6a+3a$ and $3a+6a$. Of $4P+7P+2P$ and $7P+2P+4P$, or $2P+4P+7P$.

30. Compare the values of $5+2+6$ and $5+(2+6)$. Of $6H+4H+2H$ and $6H+(4H+2H)$.

Combine the similar terms in the following:

31. $3a+2a+4+7$.

SOLUTION. $3a+2a=5a$; $4+7=11$.

Hence, $3a+2a+4+7=5a+11$.

32. $5N-2N+9-3$.

36. $10m-8+10-8m$.

33. $12z+4+6z+2$.

37. $3F-12-F+20$.

34. $9R-3+2R+7$.

38. $K+10+7K-3-4K-6$

35. $20+v+5+4v$.

39. $12s+18-s-12-5s$.

SUPPLEMENTARY EXERCISES

Evaluate the following

1. $A=lw$, when $l=16$ and $w=14$.

2. $A=\frac{1}{2}bh$, when $b=18$ and $h=14\frac{1}{2}$.

3. $C=\pi d$, when $d=16$ and $\pi=3\frac{1}{7}$.*

4. $C=2\pi r$, when $r=12$, and $\pi=3\frac{1}{7}$.

5. $A=\pi r^2$, when $r=9$ and $\pi=3.1416$.

6. $A=\frac{1}{4}\pi d^2$, when $d=8$ and $\pi=3.1416$.

7. $A=4\pi r^2$, when $r=16$, and $\pi=3\frac{1}{7}$.

8. $A=\pi d^2$, when $d=25$ and $\pi=3.1416$.

9. $V=\frac{1}{3}\pi r^2h$, when $r=6$, $h=12$, and $\pi=3.1416$.

10. $V=\frac{4}{3}\pi r^3$, when $r=8$ and $\pi=3.1416$.

11. $A=\frac{1}{2}h(a+b)$, when $h=8$, $a=10$, and $b=15$.

12. $V=\frac{1}{3}h(b+B+\sqrt{bB})$, when $h=16$, $b=25$, and $B=49$.

13. $V=\frac{1}{3}\pi h(r^2+r'^2+rr')$, when $\pi=3\frac{1}{7}$, $h=10$, $r=6$, and $r'=9$.

14. Recall the rules you studied in arithmetic, and give the meanings of as many of the above formulæ as possible.

* In practical work $3\frac{1}{7}$, instead of 3.1416, is often used for the value of π .

In the formula for simple interest, $i = prt$, find i when:

15. $p = \$500$, $r = 5\%$, $t = 2$. 18. $p = \$900$, $r = 4\frac{1}{2}\%$, $t = 3$.
 16. $p = \$1650$, $r = 6\%$, $t = 3\frac{1}{2}$. 19. $p = \$1200$, $r = 5\%$, $t = 1\frac{2}{3}$.
 17. $p = \$780$, $r = 5\frac{1}{2}\%$, $t = 2\frac{1}{2}$. 20. $p = \$1500$, $r = 5\%$, $t = \frac{3}{4}$.

21. The sum to which p dollars placed at compound interest will amount in t years at the rate per cent r is $p(1 + r)^t$.

To what sum will \$500 amount if placed at compound interest for 2 years at 6%? For 3 years?

22. To what sum will \$2000 amount if placed at compound interest for 3 years at 5%?

23. I invest \$100 in the stock of a building and loan association which pays 7% interest compounded annually. To what will it amount in 3 years?

24. The number of ways that a committee of 3 persons may be selected from a group of n persons is $\frac{n(n-1)(n-2)}{6}$. In how many ways may a committee of 3 be selected from 4 persons? From 5 persons? From 6 persons?

25. If n gymnastic exercises may be taken in any order, the total number of different ways in which they may be selected to follow each other is $n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1$. In how many different ways may a series of 5 exercises be arranged? 6 exercises? 7 exercises?

NOTE. — The expression $n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1$ means for any number n , the product of this number and each consecutive number to 1. Thus, if $n = 4$, the expression becomes $4 \times 3 \times 2 \times 1$.

26. A party of 8 people secure a row of seats at a theater. In how many different ways may they be seated? (See Problem 25.)

27. The area of an equilateral triangle, each of whose sides is s , is found by the formula $A = \frac{1}{4}s^2\sqrt{3}$. Find A when $s = 8$. ($\sqrt{3} = 1.732$).

28. By drawing the figure of a race track with two straight parallel sides S , and with semicircular ends each with radius r ,

you will see from your knowledge of arithmetic that the distance, D , around the track is expressed by: $D = 2(S + \pi r)$. Find D when $S = \frac{1}{4}$ mi. and $r = \frac{1}{8}$ mi.



29. In a figure (polygon) with n equal angles, the number of degrees in each angle is $\frac{180(n-2)}{n}$. How many degrees in each of the equal angles of a triangle? Of a figure with 5 equal angles (pentagon)? Of a figure with 6 equal angles (hexagon)? Of a figure with 12 equal angles?

SUPPLEMENTARY VOCATIONAL FORMULÆ

NOTE.—The following exercises are added for those who wish to give a more extensive course in the *evaluation of formulæ*. It is not intended that the teacher should make any attempt to explain the meaning of any of them. The uses of the various formulæ are stated in the belief that students will be more interested in the process of evaluation if they know that such a thing must be done by people doing the world's work. *They may be omitted without interfering with the subsequent work.*

1. The formula $C = \frac{E}{R}$ is much used in work with electricity. Compute C when $E = 20.5$ and $R = 16.75$.

2. The velocity of the recoil of guns is computed by the formula $V = \frac{wv}{W}$, where V = velocity of recoil, W = weight of gun and carriage, in pounds, w = weight of projectile, and v = muzzle velocity of projectile. A 10-inch gun on a battleship fires a 400-pound projectile with a muzzle velocity of 1600 feet per second. Weight of gun and carriage, 22 tons. Find velocity of recoil.

3. The force of pressure P of the wind, in pounds per square foot, is computed from $P = .005 V^2$, where V = velocity of wind in miles per hour. Find the force of the wind when blowing at 40 miles an hour. What would be the total pressure of this wind against the side of a house 20 feet high and 60 feet long?

4. If an object, such as a brick dislodged from the wall, starts from rest and falls towards the earth, the distance that it will fall in a given length of time is computed by the formula $s = \frac{1}{2}at^2$, where s = distance in feet, $a = 32$, and t = number of seconds elapsed. Find the distance an object will fall in 1 second; 2 seconds; 3 seconds; 4 seconds; 10 seconds; 60 seconds; 5 minutes.

NOTE.—This formula holds accurately only for bodies falling in a perfect vacuum. For bodies falling through the air, the velocity is somewhat diminished by the resistance of the air.

5. How far would a body fall during the sixth second?

SUGGESTION.—Find the distance it would fall in 5 seconds and in 6 seconds.

6. How far would a falling body move during the thirtieth second?

7. To measure temperature, two different kinds of thermometers are in use: the Fahrenheit and the Centigrade. On the former the freezing point is marked 32° and the boiling point 212° . On the latter these are marked 0° and 100° respectively. If the temperature is read on a Fahrenheit thermometer, the corresponding temperature on the Centigrade thermometer is computed by the formula $C = \frac{5}{9}(F - 32)$, where C = temperature in degrees on Centigrade scale and F = temperature in degrees on Fahrenheit scale.

When it is 70° by the Fahrenheit thermometer, what is the temperature on the Centigrade thermometer? When 64° ? When 48° ? When 80° ?

8. The strength or capacity for work of engines is expressed by horse power. The horse power of steam engines is found by the formula

H.P. = $\frac{plan}{33000}$, where p = pressure of steam in

pounds per square inch, l = length of stroke in feet, a = area of piston in square inches, and n = twice the number of revolutions



per minute. Compute the horse power of an engine in which a test shows $p = 95$ pounds, $l = 30$ inches, $a = 706.8$ square inches, and $n = 100$.

9. Find the horse power of a steam engine in which $p = 110$ pounds, $l = 24$ inches, $n = 120$, and the diameter of the piston is 16 inches.

10. The horse power of automobile engines is computed by the formula $H. P. = KND(D - 1)(R + 2)$, where $K = .197$ for commercial touring cars, $N =$ number of cylinders, $D =$ diameter of cylinders, and $R =$ ratio of the stroke to the diameter. What is the horse power of a 4-cylinder engine of a touring car in which the diameter is 4 inches and the stroke 5 inches?

11. If a beam L feet long is supported at both ends and loaded uniformly throughout its length with W pounds per foot, the greatest bend or deflection D at the middle, in inches, is obtained from



the formula $D = \frac{5 WL}{384 EI}$, where E and

I have particular values depending upon the material used. This formula is used by architects in designing buildings. If a beam in which $E = 30,000,000$ and $I = \frac{5}{8}$ is 12 feet long, and the load 200 pounds per foot, find the deflection.

12. The elevation of a point above sea level is obtained by use of the thermometer from the formula $H = 513t + t^2$, where $H =$ height in feet above sea level, and $t =$ difference (in degrees Fahrenheit) between 212° and the temperature at which water boils at the place of observation.

The temperature of boiling water at a certain place is 210° . Find the elevation of the place.

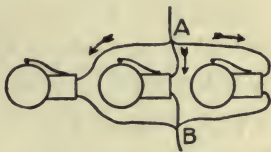
13. The relation between the height of a chimney and the pressure of draft which it produces is given by the formula $P = H \left(\frac{7.6}{460 + T} - \frac{7.9}{460 + t} \right)$, where $P =$ pressure of draft as measured by the height in



inches of a column of water that it will support in a tube, H = height of chimney in feet, T = temperature outside, and t = temperature of air in chimney.

If $H = 150$ feet, $T = 50^\circ$, and $t = 600^\circ$, find P .

14. In electrical work problems of the following kind are encountered: An electric current flowing from a point A to a point B is divided at A into three branches, each passing through an electric bell, and the branches are united again into one current at B . The total resistance R of the circuit from A to B is computed by the

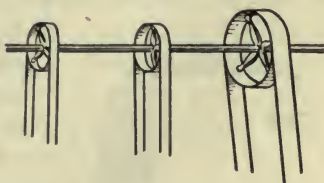


formula $R = \frac{r_1 r_2 r_3}{r_1 r_2 + r_1 r_3 + r_2 r_3}$, where r_1 , r_2 , and r_3 are the respective resistances of the three branches. If $r_1 = 1.2$ ohms, $r_2 = 1.4$ ohms, and $r_3 = 1.6$ ohms, find the total resistance R .

NOTE.—Problem 14 shows how the system of representing numbers by letters may be extended by attaching *subscripts* to the letters of the alphabet. Thus, by attaching subscripts to the letter a we can create any number of new symbols for representing numbers, as a_1 (read a sub 1), a_2 , a_3 , a_4 , etc. The symbols a_1 and a_2 represent distinct and unrelated values, just as a and b do.

Similarly, by attaching *superscripts* to letters, new number symbols may be formed, as x' (read x prime), x'' (read x second), x''' , x^{iv} , x^v , etc. Use is sometimes made of letters of the Greek alphabet, α (alpha), β (beta), γ (gamma), δ (delta), etc.

15. The horse power that may be transmitted safely by a certain kind of shafting without breaking or twisting is computed by the formula $H. P. = \frac{nd^3}{64}$, where $n =$



number of revolutions per minute, and d = diameter of shafting in inches. How many horse power can be transmitted by such a shafting of 4 inch diameter, making 76 revolutions per minute? By one of 5 inch diameter making 100 revolutions per minute?

16. When a brick arch is supported by a tie rod to keep the walls from spreading, the "horizontal thrust," or stretching force exerted on the rod, in pounds per linear foot of arch, is obtained by the formula

$$F = \frac{1.5 WS^2}{R}, \text{ where } W = \text{weight on arch}$$

in pounds per square foot, S = span of

arch in feet, R = rise of arch in inches. Find the strain on the tie rod in an arch on which the weight is 360 pounds per square foot, the span 4 feet, and rise of arch 18 inches.



17. The discharge of a pump in gallons per minute is obtained from the formula $G = .03264 Td^2$, where G = number of gallons, T = travel (total distance traveled) of piston in feet per minute, d = diameter of cylinder in inches. Suppose that the diameter of the cylinder of a pump is 18 inches, that the stroke of the piston is 24 inches, and that it makes 40 revolutions per minute. Find the discharge.

18. The horse power of the pump required in Problem 17 is $\text{H.P.} = .00001238 Td^2h$, where h = vertical distance in feet between levels of water at source and point of discharge. Find the horse power of the pump in Problem 17 required to raise the water to a standpipe through a height of 216 feet.

19. By the *specific gravity* of a solid substance, such as iron, is meant the ratio of the weight of that substance to the weight of an equal volume of water. The specific gravity of an object may be found by first weighing it in air, then weighing it again when suspended under water. An object seems to lose weight when weighed in water due to the buoyancy of the water.



If the specific gravity is s , then $s = \frac{W}{W - w}$, where W = weight in air and w = weight in water.

A piece of glass weighing 40 grams in air weighs 24 grams in water. Find its specific gravity.

CHAPTER II

THE EQUATION

15. The Equation.— In solving some kinds of problems use is made of *equations*. An **equation** is the statement that two expressions are equal or that they have the same value.

Thus, $2 \times 5 = 10$, $a + b = b + a$, and $4n - 3 = 2n + 5$ are equations.

The two expressions which are connected by the sign $=$ are called the **members** of the equation.

For example, in $3P - 2 = 4$ the expression $3P - 2$ is called the **first** or **left member**, and the 4 is called the **second** or **right member**.

In some equations the members are equal for all particular values of the general number involved. These are called **identical equations** or **identities**.

Thus, in $a^2 + 2ab + b^2 = (a + b)^2$ the members are equal *for all values whatever* that may be given to a and b . When $a = 1$ and $b = 2$, the equation becomes $9 = 9$; when $a = 2$ and $b = 2$, it becomes $16 = 16$; when $a = 5$ and $b = 1$, it becomes $36 = 36$; etc.

In other equations the members are not equal for all particular values of the general numbers involved. These are called **conditional equations** or simply **equations**.

Thus, $5y - 2 = 3y$ is true only under the condition that $y = 1$. And $3R + 4 = 10$ is true only when R is 2.

16. Problems Expressed by Equations.— In every problem solved by use of equations, the values of one or more numbers are *unknown*.

For example, in the problem, "If eggs cost 34 cents a dozen, find the cost of 5 dozen," the cost of 5 dozen is an *unknown number*.

To solve a problem is to find the values of the unknown numbers. In solving a problem algebraically it is first expressed in a sort of shorthand by an equation. The equation expresses the relation between the known and unknown numbers. An unknown number is expressed by a letter, and later the value represented by the letter in the equation is found.

EXAMPLE 1. A man's salary was increased by 10%, or $\frac{1}{10}$ of itself. After the increase he received, annually, \$1980. What was his salary before the increase?

Represent by d his salary before the increase.

Then the problem may be expressed by $d + \frac{1}{10}d = \$1980$.

EXAMPLE 2. Two men form a partnership in business in which the first invests twice as much as the second. Their profits are \$3600. What part of the profits should each receive in settlement?

Let a represent the amount the second should receive.

Then $2a$ represents the amount the first should receive.

Hence, the problem may be expressed by the equation

$$2a + a = \$3600.$$

To express a problem by an equation we take the following steps:

- (1) First read the problem to discover what numbers are unknown.
- (2) Let some letter represent one of the unknown numbers.
- (3) Then, from statements in the problem, express all the other unknown numbers in terms of this letter.
- (4) Finally, from another statement of the problem, form the equation between these and the known numbers of the problem.

ORAL EXERCISES

1. If n denotes a certain number, what will denote a number 10 larger? What will denote one 10 less? One 10 times as large? One $\frac{1}{10}$ as large?

2. The rainfall last year at a certain place was 6 inches more than on the year before. If x represents last year's rainfall, what will denote the rainfall of the year before? If x represents the rainfall of the year before, what will denote that of last year?

3. The melting temperature of glass is 136 degrees lower than 3 times that of zinc. If t represents the melting temperature of zinc, what will represent that of glass? $5t - 136$

4. One boy sold 20 more than $\frac{1}{2}$ as many papers as another. By use of some letter represent the number sold by each.

5. If A has $\$y$, B $\frac{2}{3}$ as much, and C $\frac{1}{2}$ as much, what will represent the amount that all three have?

6. Of two candidates at an election one was defeated by 362 votes. By some letter express the number of votes received by each.

7. A rectangle is 2 inches more than 3 times as long as it is wide. By use of a letter represent both the length and the width.

8. A merchant sold coffee at a profit of 20%. If it cost C cents a pound, what will express the selling price?

9. If I lend $\$D$ at 5% for 4 years, what will express the amount at the end of that time?

10. The distance from Chicago to New York by rail is about 900 miles. If a train ran an average of 40 miles an hour, how long would it require to make the run between the two cities? If it ran V miles an hour, what will express the time required for the run?

In each of the above problems one number has been expressed in terms of one or more other numbers.

WRITTEN EXERCISES

Express the following problems by means of equations:

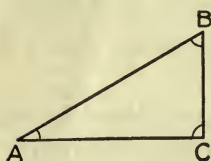
1. One of two partners in a business invests twice as much as the other. How should a profit of $\$1200$ be divided?

2. A real estate dealer sold a lot for $\$1500$, and thereby made a profit of 25%. What did it cost him?

3. It requires 2240 feet of wire fencing to inclose a rectangular piece of ground that is three times as long as it is wide. Find its width and length.

4. If I think of a number, double it, and add 8 to the result, I get 40. What is the number?

5. In the latitude of Chicago, on the longest day of the year, the day is 6 hr. 8 min. longer than the night. What is the length of each?



6. In the triangle ABC , the sum of the three angles is 180° . Angle B is twice as large as angle A , and angle C is three times as large as angle A . How many degrees in each angle?

17. Solving an Equation.—It has been seen that a problem may be expressed by an equation, in which the unknown numbers are represented by use of a letter. It is evident that the problem may be solved if the value of the letter in the equation may be found. This is called *solving the equation*.

To solve an equation is to find the particular value or values of the unknown number that make the two members equal. A particular value of the unknown number thus found is called a *root* of the equation.

Thus, in $4y + 6 = 2y + 10$, if y is 2, each member equals 14. Hence, 2 is a *root*. The equation is said to “be satisfied” when $y = 2$.

Before proceeding to the solution of problems, we must discover how to solve an equation.

18. Axioms.—If the weights on the two pans of a balance are equal, they balance. If equal weights are added to, or taken from, the two pans, will the resulting weights balance? If the weights in the two pans are made twice as great, three times as great, etc., or one half as great, one third as great, etc., will the resulting weights balance?



These facts about the balance illustrate certain general principles in dealing with numbers which may be assumed. They are called **axioms**, and may be stated as follows:

1. *If equal numbers are added to equal numbers, the sums are equal.*
2. *If equal numbers are subtracted from equal numbers, the remainders are equal.*
3. *If equal numbers are multiplied by equal numbers, the products are equal.*
4. *If equal numbers are divided by equal numbers (not zero), the quotients are equal.*

An equation may be compared to a balance. The members of the equation correspond to the weights on the two pans of the balance. In the figure, the fact that the weights balance is expressed by the equation $P + 4 = 9$. Just as the weights on the two pans may be equally increased, decreased, multiplied, or divided, and continue to balance, so the members of an equation may be equally increased, decreased, multiplied, or divided, according to the above axioms, and remain equal.



The use of these axioms in solving equations is shown in the following sections.

19. Equations Solved by a Single Addition, Subtraction, Multiplication, or Division.

EXAMPLE 1. — Solve $W + 6 = 10$.

Since $W + 6$ is 6 greater than W , what must be done to $W + 6$ to get W ?
Subtracting 6 from each member, $W = 4$. Axiom 2.

EXAMPLE 2. — Solve $8a = 7a + 5$.

What must be done to $7a + 5$ to get 5?

Subtracting $7a$ from each member, $a = 5$.

Axiom 2.

EXAMPLE 3. — Solve $S - 4 = 12$.

Since $S - 4$ is 4 less than S , what must be done to $S - 4$ to get S ?

Adding 4 to each member, $S = 16$.

Axiom 1.

EXAMPLE 4. — Solve $3x = 24$.

What must be done to $3x$ to get x ?

Dividing each member by 3, $x = 8$.

Axiom 4

EXAMPLE 5. — Solve $\frac{n}{6} = 4$.

By what must $\frac{n}{6}$ be multiplied to get n ?

Multiplying each member by 6, $n = 24$.

Axiom 3.

ORAL EXERCISES

Solve:

- | | | |
|----------------------|--|-----------------------------|
| 1. $a + 3 = 7$. | 13. $p - 9 = 1$. | 25. $12\frac{1}{2}c = 25$. |
| 2. $P + 6 = 20$. | 14. $R - 2 = 6$. | 26. $\frac{y}{4} = 25$. |
| 3. $v + 1 = 8$. | 15. $w - 14 = 16$. | 27. $\frac{n}{6} = 3$. |
| 4. $R + 9 = 16$. | 16. $x - \frac{1}{2} = \frac{1}{2}$. | 28. $\frac{A}{12} = 4$. |
| 5. $3x = 2x + 4$. | 17. $m - 1\frac{1}{3} = \frac{2}{3}$. | 29. $\frac{r}{10} = 7$. |
| 6. $6A = 5A + 12$. | 18. $5y = 15$. | 30. $\frac{Q}{5} = 8$. |
| 7. $8k = 23 + 7k$. | 19. $4H = 24$. | 31. $\frac{V}{20} = 18$. |
| 8. $9t = 3 + 8t$. | 20. $2a = 18$. | |
| 9. $16D = 5 + 15D$. | 21. $6Q = 48$. | |
| 10. $B - 4 = 20$. | 22. $7S = 28$. | |
| 11. $x - 10 = 15$. | 23. $24D = 72$. | |
| 12. $M - 50 = 12$. | 24. $100z = 125$. | |

WRITTEN EXERCISES

Solve:

- | | |
|--|---|
| 1. $x + 3.25 = 5$. | 9. $t - 7\frac{3}{7} = 8\frac{1}{8}$. |
| 2. $D - 0.426 = 4.32$. | 10. $y - \frac{3}{16} = \frac{5}{12}$. |
| 3. $4.6A = 3.6A + 2.5$. | 11. $1\frac{1}{2}r' = 6$. |
| 4. $P - 4.08 = 8$. | 12. $2\frac{3}{8}v = 5\frac{3}{8}$. |
| 5. $2.16m = .24$. | 13. $\frac{W}{3\frac{1}{4}} = 4\frac{1}{3}$. |
| 6. $3.1416d = 9.5$. | 14. $\frac{D}{3\frac{5}{6}} = 4\frac{2}{3}$. |
| 7. $4\frac{1}{2}C = 7 + 3\frac{1}{2}C$. | |
| 8. $H + 3\frac{3}{8} = 7\frac{1}{16}$. | |

20. Any Simple Equation Solved. — Any simple equation may be solved by using one or more of the processes employed in § 19, as shown by the following examples.

EXAMPLE 1. — Solve $4n + 15 + 13n = 5n + 99$.

Adding similar terms, $4n$ and $13n$, $17n + 15 = 5n + 99$.

Subtracting 15 and $5n$ from each member, $12n = 84$.

Dividing each member by 12, $n = 7$.

Notice that by subtracting 15 from each member we get the term free of n out of the first member, and by subtracting $5n$ from each member we get the term containing n out of the second member.

Check. — When $n = 7$, each member becomes 134, which shows that the answer is correct.

A number symbol put in place of another in an expression, as in checking the answer in the preceding example, is said to be **substituted** for it. The student is familiar with substitution in the formulæ in Chapter I.

The root of every equation should be checked, or tested for accuracy, by substituting the value found in place of the unknown number in each member of the equation.

EXAMPLE 2. — Solve $7V - 8 + 6V = 28 + 4V$.

By combining similar terms, what does the first member become?

What must be added to each member to free the first member of the term not containing V ?

What must be subtracted from each member to free the second member of the term containing V ?

Show that the resulting equation is $9V = 36$.

By what must each member now be divided?

Check the answer by substituting its value in each member of the given equation and seeing if they are equal.

EXAMPLE 3. — Solve $\frac{2x}{3} + \frac{3x}{4} + 9 = \frac{5x}{6} + 16$.

What is the least number divisible by each denominator?

By what, then, must each member be multiplied to free each term of fractions.

Show that the resulting equation is $8x + 9x + 108 = 10x + 192$.

Now show that $x = 12$.

Check by substituting 12 for x in the given equation.

It is evident from these examples that the steps in solving such equations are as follows :

(1) *If the equation contains one or more fractions, multiply both members by the L. C. M. of all the denominators to free the equation of fractions.*

(2) *Unite similar terms by adding, or subtracting, as the case may require.*

(3) *Free the first member of all terms that do not contain the unknown number, and the second member of all terms that do contain the unknown number, by adding the same number to both members or subtracting the same number from both members. To remove a term with a plus sign before it, subtract it, and to remove a term with a minus sign before it, add it to both members.*

(4) *Divide both members of the resulting equation by the coefficient of the unknown number.*

(5) *Check the work by substituting the value of the root found in each member of the given equation.*

EXERCISES

Solve:

1. $7a + 3 = a + 21.$

9. $6M - 5 = 4M + 1.$

2. $4b + 4 = 25 + b.$

10. $5y + 5 = 8 - 3y.$

3. $5P - 4 = 12 - 3P.$

11. $20R - 25 = 5R + 5.$

4. $2v - 5 = 7 - v.$

12. $13h + 15 = 11h + 35.$

5. $3D - 7 = 14 - 4D.$

13. $4z - 3 = z.$

6. $5x + 3 = 3x + 9.$

14. $12c - 13 + c = 35 + 7c.$

7. $200A - 50 = 50A + 250.$

15. $13V - 7 = 5V + 2 - V.$

8. $5k - 5 = k + 3.$

16. $3x = 10 + x.$

17. What is the least number divisible by both 2 and 3? By what one number may both of the fractions $\frac{x}{2}$ and $\frac{x}{3}$ be multiplied to change them to whole numbers?

18. By what one number may each of the fractions $\frac{2a}{3}$, $\frac{a}{4}$, and $\frac{5a}{6}$ be multiplied to change all of them to whole numbers?

19. What is 6 times 4 apples? 6 times $4a$? 3 times $5x$? 8 times $7n$? 10 times $3A$?

20. What is 6 times $\frac{2}{3}$? 8 times $\frac{3}{4}$? 10 times $\frac{3x}{5}$? 12 times $\frac{3t}{4}$?

Solve:

$$21. \frac{t}{4} + \frac{t}{5} = \frac{t}{10} + 7.$$

$$26. s + \frac{s}{2} + \frac{s}{3} = \frac{s}{6} + 30.$$

$$22. \frac{3n}{4} = \frac{2n}{3} + 15.$$

$$27. x - \frac{x}{2} = 30 + \frac{x}{6}.$$

$$23. \frac{2x}{3} + \frac{3x}{4} = \frac{5x}{6} + 7.$$

$$28. 3W - \frac{W}{3} = \frac{W}{2} + 13.$$

$$24. \frac{a}{2} + 6 = 17 - \frac{3a}{5}.$$

$$29. 10 - \frac{k}{8} = \frac{k}{4} + \frac{k}{2} + 3.$$

$$25. 2B + \frac{5B}{6} - 17 = 0.$$

$$30. R - 10 + \frac{R}{7} = \frac{R}{2} - \frac{R}{14}.$$

21. Problems Solved by Equations.—The following example shows the complete process of solving a problem by use of an equation.

EXAMPLE.—At an election two candidates, A and B , together received 2245 votes. A was elected by a majority of 285 votes. How many votes were cast for each?

Let n = number cast for A .

Then $n - 285$ = number cast for B .

Hence, $n + n - 285 = 2245.$

Solving, $2n - 285 = 2245.$

$$2n = 2530.$$

$$n = 1265, \text{ number cast for } A.$$

Hence, $n - 285 = 980, \text{ number cast for } B.$

EXERCISES

1. The initiation fee of a certain organization is \$40, and the annual dues \$5. In how many years will a member have paid into the treasury \$100?

2. It costs \$4 to have made the plates for printing a circular, and the cost of paper and printing is $\frac{1}{4}$ cent a copy. How many copies can I have made for \$8?

3. One partner has 4 times as much money invested in an enterprise as the other. How should a profit of \$2500 be divided?

4. Two boys agree to mow a man's lawn during a season for \$6. One mows it 3 times and the other 5 times. How should the money be divided between them?

5. A grocer bought some eggs at 16 cents a dozen. Thirty were broken, and he sold the remainder at 18 cents a dozen. He found that he got back just the cost of the whole lot. How many did he buy?

6. Divide \$75 between A and B so that A shall receive \$3 more than twice as much as B.

7. Two boys sold 45 papers. One sold $\frac{4}{5}$ as many as the other. How many did each boy sell?

8. A boy has \$1.05 in dimes and nickels. He has just as many dimes as nickels. How many has he of each?

9. Two trains start at the same time from stations 450 miles apart, one running at the rate of 35 miles per hour, and the other at the rate of 40 miles per hour. In how many hours do they meet?

10. The resistance of an electrical battery and the wire attached to it is found to be 3 ohms. By connecting to the battery a wire of the same size and material and three times as long, the resistance is found to be 6 ohms. Find the resistance of the battery.

11. The resistance of a battery and the wire attached to it is $5\frac{1}{2}$ ohms. When a wire of the same kind and 4 times as long is

connected to the battery, the resistance is found to be 16 ohms. Find the resistance of the battery.

12. The total number of admission tickets to a circus was 836. The number of tickets sold for adults was 136 less than twice the number of children's tickets. How many were sold of each?

13. The distance from Chicago to San Francisco by rail is 2563 miles, which is 397 miles more than twice as far as from Chicago to Denver. How far is it from Chicago to Denver?

14. In a class containing 24 boys and girls there are 6 more girls than boys. How many of each are there?

15. It takes 360 feet of wire fencing to inclose a rectangular lot 60 feet wide. How long is the lot?

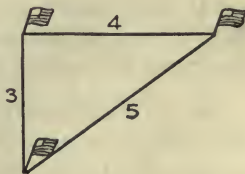
16. A boat steams against the current of a stream at a speed of 3 miles per hour, and with the current at a speed of 15 miles per hour. Find the speed of the current.

17. A train left a station and traveled at the rate of 30 miles per hour, and three hours later a second train left the station, following on the same track at the rate of 40 miles per hour. In how many hours from the time that the first train started was it overtaken by the second train?

18. A bar 40 inches long is to be cut into two pieces. The shorter piece is to be $\frac{2}{3}$ as long as the longer piece. Find the lengths of the pieces.

19. A board 62 inches long is to be sawed into two pieces so that one piece is 8 inches longer than the other. How long will each piece be?

20. Surveyors, in order to mark off a right angle on the ground, sometimes set three stakes so that when a tapeline is stretched around them so as to form a triangle, it is divided at the stakes into parts whose lengths are as 3, 4, and 5. The ancient Egyptians used this method in laying out their pyramids, etc. If a 100-foot tapeline is



used, find the lengths of the parts into which it must be divided at the stakes.

SUGGESTION. — Let $3n$ represent the number of feet in the shortest part.

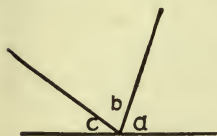
21. If, in Problem 20, the surveyors use a 50-foot tapeline, find the lengths of the parts into which it must be divided at the stakes.

22. A surveyor's chain which contains 60 links is divided into three parts whose lengths are as 3, 4, and 5, and used instead of the tapeline in Problem 20. How many links in each part?

NOTE. — Students who have time will find it interesting to use the method of Problems 20-22 in making out-of-door measurements.

23. A merchant sells tea for 70 cents a pound, and thereby makes a profit of 40% on the cost. What did it cost him?

24. How much money must I invest in a business yielding 20% profit in order that the investment and profit together may amount to \$1000?



25. The sum of the angles a , b , and c is 180° . If angle b equals angle a , and angle c equals one-half of angle a , find the value of each.

26. The sum of the angles of a triangle is 180° . If two of the angles are equal, and the other one twice as large as either of them, how many degrees in each?

27. An angle of 75° is made up of the sum of an angle of 15° and 5 equal angles. How many degrees in each of the equal angles?

28. Two angles are called complementary when their sum is 90° . If one of two complementary angles is 15° more than twice the other, how many degrees in each?

29. Three angles just cover all of the plane around a point. The difference between the second and the first is 60° and between the third and the second 60° . How many degrees in each?

30. If a polygon has n sides, the sum of all of its angles is $180^\circ n - 360^\circ$. How many sides has a polygon the sum of whose angles is 720° ?



31. Whole numbers such as 5 and 6 are consecutive whole numbers. The sum of two consecutive whole numbers is 71. What are the numbers?

32. The sum of two consecutive odd numbers is 32. What are the numbers?

33. One number is twice as large as another. If I take 4 from the smaller and 16 from the larger, the remainders are equal. What are the numbers?

34. A number, its half, its third, and its fourth make 100. Find the number.

35. In the papyrus written by Ahmes, the Egyptian, about 1700 B.C., the unknown number was called "hau." This ancient book contained the following problem: "Hau, its $\frac{1}{7}$, its whole, it makes 19;" i.e. $\frac{1}{7}$ of the unknown number and the unknown number make 19. What was the unknown number?

SUPPLEMENTARY EXERCISES

1. A certain number exceeds n times the number x by m . What will express the number?

2. D dollars are invested at p per cent interest. What will represent the sum to which this will amount in 1 year? In 3 years? In t years?

3. A merchant sells sugar costing c cents a pound at a profit of 12%. What will represent the selling price? If it sells at a profit of p %, what will represent the selling price?

4. Solve $\frac{s}{12} + \frac{s}{8} = 2\frac{1}{2}$.

5. Solve $\frac{3F}{4} + \frac{2F}{5} = \frac{5F}{6} + 6\frac{1}{3}$.

6. Solve $7 = \frac{y}{6} - \frac{2y}{9} + \frac{y}{4}$.

7. I paid \$6 for an advertisement of 6 lines, as follows: 20 cents a line for the first insertion, 10 cents a line for each of the next 6 insertions, and 2 cents a line after that. Find the number of insertions at 2 cents a line.

8. How much per bushel must a merchant pay farmers for wheat in order to market it at \$1.14 and make a profit of 4 % ?

9. How much per bushel must a wholesale house pay for potatoes in order to sell them to retailers at 53 cents and make a profit of 6 % ?

10. How much must a country merchant pay farmers for turkeys in order to make 6 %, if he can get 17 cents per pound for them in the city market?

11. In the city market, on a certain date, unwashed wool sold for 25 cents per pound. How much per pound must a wool buyer pay farmers for their unwashed wool in order to make a profit of 10 % when selling it at this price, if it costs him 3 cents per pound to market it ?

12. On a certain date hogs sold for \$9.75 per hundred pounds in the Chicago market. What must a buyer pay for hogs in the country in order to market them at this price and make a profit of 20 %, making no allowance for cost of delivery ?

13. At what market price must 1-year 4 % bonds be offered for sale, in order that the buyer, by holding them until maturity, may make 8 % on his investment?

SUGGESTION. — The profit made must come from two sources, the interest on the par value, which is \$4, and the excess of the maturity value over the price paid for the bonds. If x is the price paid, $100 - x + 4 = .08x$.

14. At what market price must 1-year 7 % bonds be purchased, if by holding them until maturity the purchaser makes 10 % on his investment?

15. At what price must 3-year 5% bonds (bonds that mature in 3 years) be purchased, if by holding them until maturity the purchaser makes 6% on his investment?

SUGGESTION. — If x is the cost required, the annual profit due to excess of maturity value over cost is $\frac{100 - x}{3}$.*

16. At what price were Lake Shore 4's bought in 1910, which, if held until maturity in 1931, would earn the buyer $4\frac{2}{3}\%$ on his investment?

17. Lackawanna 5's bought in 1910 and maturing in 1923, if held until maturity, would pay the purchaser $5\frac{9}{10}\%$ on his investment. What was the price of Lackawanna 5's when bought?

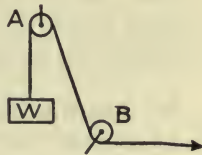
18. Chicago & N. W. stock bought May 13, 1913, and sold May 13, 1913, at 173, after paying three 12% dividends, would yield the purchaser $12\frac{1}{2}\%$ on his investment. At what was the stock bought May 13, 1913?

SUGGESTION. — If n = amount paid for a \$100 share, show that

$$\frac{173 - n}{3} + 12 = .125n.$$

19. A merchant sold an article at a gain of 40%. When the cost increases \$90, the same selling price will be a loss of 50%. Find the cost.

20. A weight W is lifted by means of a rope running over two pulleys, A and B . The friction in each pulley increases the tension in the rope 2%. Find W if the force required to lift it is 400 pounds.



* This method of distributing the loss or gain at maturity does not consider interest upon the annual loss or gain. For a full discussion, see the *Stone-Millis Secondary Arithmetic*, page 90.

CHAPTER III

POSITIVE AND NEGATIVE NUMBERS

22. Extension of the Number System. — We have seen in Chapter I a class of practical problems which are solved by use of the *formula*, and which require a knowledge of *literal notation*. In Chapter II we have seen another class of problems to solve which requires a knowledge of the *equation*. There are still other problems, now to be considered, which require us to extend our idea of number to include a *new kind of number*.

If a man deposits \$500 in a bank, and then withdraws \$300, the amount of his balance is $\$500 - \$300$, or \$200. It sometimes happens that a depositor *withdraws more money than he has deposited*, or makes an overdraft. For example, suppose that he has deposited only \$250, and withdraws \$300. There is now a balance of \$50 against him, showing that he owes the bank and not that the bank owes him. The balance is obtained by subtracting \$300, the amount withdrawn, from \$250; *i.e.* the balance is $\$250 - \$300$.

This problem, then, requires us *to subtract a number from a smaller number*. In order to solve such problems, our idea of number must be extended and this subtraction made possible.

23. Negative Number. — The remainder resulting from subtracting a number from a smaller number is called a **negative number**. We must now see how to represent a negative number.

If a person wishes to withdraw \$100 from a bank, he may withdraw part at a time. Thus, he may withdraw \$75 with one check and then \$25 with another. What are some ways that he may withdraw \$250 from a bank, by withdrawing only part at a time?

Similarly, in general, one number may be subtracted from another by separating the subtrahend into two or more parts, and subtracting the parts one at a time.

Hence, in attempting to subtract 7 from 4, by separating 7 into 4 and 3, we get

$$4 - 7 = 4 - 4 - 3 = 0 - 3.$$

In practice the zero is dropped, and the remainder $0 - 3$ written simply -3 .

Similarly, $5 - 9 = 5 - 5 - 4 = 0 - 4$ or -4 .

$$8 - 14 = 8 - 8 - 6 = 0 - 6 \text{ or } -6.$$

Hence, a *negative number* is expressed by a number preceded by the *minus sign*.

It is clear that a negative number, such as $-\$25$, indicates a **reserved subtraction**, there being nothing, when it stands alone, from which to subtract it. Thus, in $\$100 - \$150 = \$100 - \$100 - \$50 = -\$50$, the remainder, $-\$50$, is a part of the subtrahend that is not subtracted because there is nothing from which to subtract it.

Hence, a negative number is in nature a subtrahend.

24. Positive Numbers.—For the sake of distinguishing from negative numbers, the arithmetical numbers with which we have always dealt up to this time are called **positive numbers**. All numbers with which we are familiar, therefore, are to be classified as either *positive* or *negative* numbers.

25. Use of Signs.—For distinction in writing positive and negative numbers, the positive or arithmetical numbers are often preceded by the sign $+$ when standing alone.

Thus, 6 is written $+6$; a is written $+a$.

When clearness would not be sacrificed, however, the sign $+$ may be omitted from a positive number. If no sign is written before a number, the number is *understood* to be positive.

It is clear, then, that the signs $+$ and $-$ have two uses: to indicate addition or subtraction, and to indicate the positive or negative quality or character of numbers. *They may always be considered as signs of addition or subtraction; and when convenient, as signs of quality.*

When standing alone, the positive numbers $+1$, $+2$, etc., are read either "plus 1," "plus 2," etc., or "positive 1," "positive 2," etc.; and the negative numbers -1 , -2 , etc., are read either "minus 1," "minus 2," etc., or "negative 1," "negative 2," etc.

EXERCISES

Read the following numbers:

1. 5 ; 90 ; -1 ; -12 ; x ; $-n$; $\frac{1}{2}$; $-\frac{1}{n}$; $-\frac{a}{b}$.

What negative numbers result from the following subtractions?

2. $3-4$. 3. $8-13$. 4. $10-12$. 5. $7-16$. 6. $32-40$.
 7. $\$100-\$125$. 8. $\$500-\$650$. 9. $\$400$ deposited and
 $\$475$ withdrawn. 10. $\$250$ deposited and $\$325$ withdrawn.

26. Opposite Numbers.—Since a negative number, such as -5 , always implies a *reserved subtraction*, if it is combined with a positive number, into one number, *it tends to subtract from, or destroy of, the positive number a part equal to itself*.

Thus, when 6 and -4 are combined into one number, -4 destroys 4 of the 6 , leaving 2 as the result. And when $\$10$ and $-\$15$ are combined, $-\$10$ of the $-\$15$ destroys the $\$10$, leaving $-\$5$ as the result.

Because of this neutralizing tendency of positive and negative numbers, they are sometimes called **opposite numbers**.

If two numbers such as $+8$ and -8 , which differ only in their signs, are combined, they completely neutralize each other. Such numbers are said to have the same **absolute value**. Thus, 8 is the absolute value of both $+8$ and -8 .

27. Positive and Negative Magnitudes.—As seen in the above illustrations, deposits and withdrawals of money in a bank tend to neutralize each other, and hence are positive and negative quantities, respectively. There are many other concrete quantities or magnitudes which are capable of existing in two *opposite states* or *senses* such that one tends to neutralize or destroy an

equal amount of the other. Consequently, their numerical values are represented by positive and negative numbers. Several of these are illustrated in the following exercises.

EXERCISES

1. If in one of two sales of goods I lose \$10, and in the other I make a profit of \$15, what is the net result?

Loss and profit are thus neutralizing quantities, each tending to destroy an equal part of the other.

If profit is called positive, what will loss be called?

If \$15 profit is written $+15$, what will express \$10 loss?

2. Express with $+$ and $-$ signs the following: \$25 loss; \$40 profit; \$15 loss; \$6 loss; \$20 profit.

3. If I had \$20 in my pocket, then spent \$25 for a suit of clothes, I would have left $\$20 - \$25$, or $-\$5$. What does the $-\$5$ in this case mean?

4. Express with $+$ and $-$ signs the following: \$50 credit; \$18 debt; \$25 debt; \$60 cash in hand; \$100 borrowed; \$20 loaned; \$15 spent.

5. The number of people entering a store during a certain hour was 216, and the number coming out was 250. Express the relation of these numbers by $+$ and $-$ signs. Also express the excess of the number coming out over the number entering.

6. The amount of water pumped into a tank during an hour was 1900 gallons; and during the same hour three locomotives withdrew, respectively, 675 gallons, 750 gallons, and 650 gallons. Express with $+$ and $-$ signs these quantities, and also the excess withdrawn during the hour.

7. A locomotive pulls a train with a force of 35 tons. The resistance of the train is 20 tons. If the pull of the locomotive is positive, represent by $+$ and $-$ signs the two forces.

8. A balloon lifts up with a force of 200 pounds. How much weight must be attached to keep the balloon from rising? If the lifting force is $+$, what represents the weight?

9. A balloon lifts with a force of 250 pounds, and a weight of 175 pounds is attached to it. If the weight is called positive, what will represent the lifting force? What will represent the result when the weight is attached?

10. A battleship has a displacement of 20,000 tons (sinks into the water until it displaces 20,000 tons of water). Express with + and - signs the weight of the ship and the upward pressure of the water.

11. A diver who weighs 165 pounds displaces 156 pounds of water (is buoyed up with a force of 156 pounds). Represent by + and - signs these forces and the resultant force with which he tends to sink to the bottom.

12. If the thermometer is 5° above zero at noon, and falls 15° by night, what is the temperature by night? $5^{\circ} - 15^{\circ} = \text{what?}$ How else may 10° below zero be written? How would temperature above zero be written?

13. On Jan. 6, 1910, the Weather Bureau reported the temperature at several different places as follows: Chicago, -2° ; Davenport, -18° ; Des Moines, -12° ; Kansas City, -2° ; Madison, -14° ; Milwaukee, -8° ; Peoria, -6° ; Pueblo, -6° ; St. Paul, -16° . Were these temperatures above or below zero? What would they be if the signs were +?

14. Certain railroad stock that sold at 7 above par in January fell to 3 below par in June. Since the value of the stock was reduced 10 points, how may the 3 below par be represented algebraically?

15. Represent by + and - signs the following prices of stock which show the change in 6 months: January, 10 below par; February, 6 below par; March, 1 below par; April, 5 above par; May, 8 above par; June, 2 above par.

16. Augustus Cæsar was ruler of the Roman Empire from -31 to +14. Express these dates in other terms.



17. Archimedes, the greatest mathematician of antiquity, was born about the year -287 , and was slain by a Roman soldier while studying a geometrical figure that he had drawn in the sand in -212 . Express these dates in other terms.

18. If a file of soldiers moves 10 paces to the front, then 6 paces to the rear, it advances how many paces from its first position? If it moves 6 paces to the front, then 10 paces to the rear, it advances $6 - 10$, or how many paces? If motion forward is positive, what is motion backward?

19. A switch engine, in making up a train, moves forward 100 yards, then backward 150 yards. The two are equivalent to what single motion? Represent by appropriate signs these two motions and the one equivalent motion.

20. If longitude west is positive, longitude east negative, latitude north positive, and latitude south negative, in what country is a city whose longitude is $+106^\circ$ and latitude $+40^\circ$? Longitude -16° and latitude $+48^\circ$? Longitude -40° and latitude -15° ? Consult a map.

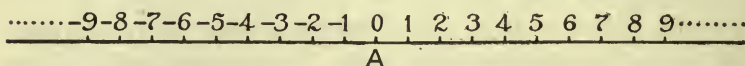
21. The longitude of the city of St. Petersburg, Russia, is given as -31° . This means that it is 31° in what direction from the Prime Meridian? Washington, D.C., is in 77° west longitude. How else may this be expressed?

22. The latitude of New York City is about $+41^\circ$. What will express the latitude of Cape Town, Africa, which is about 34° south of the equator?

28. **Numbers Represented by Distances.** — In the preceding exercises, it has been seen that many magnitudes may be represented numerically by positive and negative numbers. These include bank deposits and withdrawals, loss and gain in business, income and expense, credits and debts, increases and decreases of physical quantities, temperature above and below zero, dates in history before and after the birth of Christ, stocks above and below par, forces acting in opposite directions, motions in opposite directions,

and longitude east and west and latitude north and south. Many others might be suggested.

Since motions in opposite directions tend to neutralize each other, the *distances* moved through in opposite directions may themselves be considered as positive and negative. In representing distances as positive and negative, the signs serve to tell the *directions* in which the distances are measured, as in the case of longitude and latitude. It follows that positive and negative numbers may be represented to the eye by distances measured on



a line. *Positive* numbers are represented by distances measured to the *right* from a starting point A, and *negative* numbers by distances measured to the *left*.

EXERCISES

1. On the above line, how will -12 be represented? $+10$? $-2\frac{1}{2}$? $+3\frac{1}{2}$? -40 ? $+100$?

2. Draw a horizontal line and mark the middle point *O*. Then mark on it distances representing the following numbers, a unit being represented by a quarter of an inch: $+2$, $+5$, $+8$, $+10$, $+6\frac{1}{2}$, -3 , -4 , -7 , -12 , $-4\frac{1}{2}$, $-10\frac{1}{2}$.

3. On a horizontal line represent the following dates (*i.e.* years): -31 , beginning of Augustus Cæsar's reign; $+14$, end of Augustus Cæsar's reign; -146 , fall of the Roman Empire; $+70$, destruction of Jerusalem by Titus; -275 , death of Euclid, who wrote the first great book on geometry; -287 , birth of Archimedes, the greatest mathematician of antiquity.

4. On a vertical line, represent the following temperatures: $+5^\circ$; $+10^\circ$; $+15^\circ$; $+20^\circ$; -5° ; -10° ; -15° ; -20° .

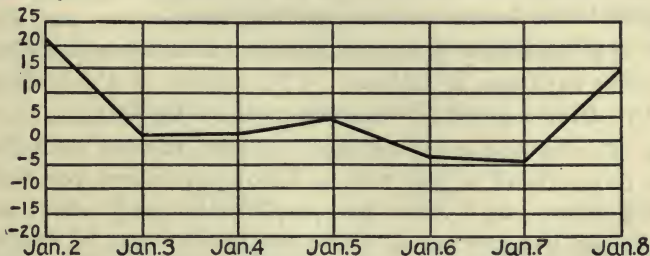
5. The minimum temperatures at St. Paul on the six days beginning Feb. 9, 1910, were $+16^\circ$, -10° , $+6^\circ$, $+4^\circ$, $+22^\circ$. Represent these on a line.

6. The longitude of New York is $+74^\circ$, of Berlin -13° , of Pekin -116° , of Calcutta -88° , of Pittsburg $+80^\circ$, of St. Louis $+90^\circ$, of Rome -12° . Draw a horizontal line and mark off on it the longitudes of these places.

29. Graphs. — The following table gives the minimum temperature at Chicago on each of the seven consecutive days beginning with Jan. 2, 1910:

DATE	JAN. 2	JAN. 3	JAN. 4	JAN. 5	JAN. 6	JAN. 7	JAN. 8
TEMP.	21°	1°	1°	4°	-4°	-5°	14°

Suppose that a sheet of paper is ruled by horizontal and vertical lines, as shown in the following diagram, each vertical line representing the thermometer scale. By marking the temperature for each date on the corresponding vertical line, and joining the consecutive points thus marked by straight lines, the heavy broken line is obtained, showing to the eye the variation from day to day.



Such diagrams as this are called **graphs**. They are much used in practical work to depict to the eye graphically the changes in quantities, such as temperature, death rates, population, imports and exports, cost of living, etc.

NOTE. — Ruled paper, often called “squared paper” or “coördinate paper,” may be obtained cheaply for drawing graphs. It is ruled accurately into small squares, and saves much time when used. This paper will be needed in other exercises later.

EXERCISES

1. The following table gives the minimum temperature at St. Paul, Minn., on each of seven consecutive days, beginning Jan. 5, 1910:

DATE	JAN. 5	JAN. 6	JAN. 7	JAN. 8	JAN. 9	JAN. 10	JAN. 11
TEMP.	6°	- 16°	- 12°	- 4°	- 12°	2°	20°

Draw a graph showing the variation in temperature during the period.

2. According to the report of the Weather Bureau, on Jan. 5, 1910, the height of the Ohio River above low-water mark at Pittsburg was 11.3 feet. The amounts in feet by which it rose or fell on the seven consecutive days thereafter were as follows, the negative sign indicating fall: - 2.1, - 2.8, 1.2, 1.3, - 2.2, - 2.7, - 1.4.

Draw a graph showing the rise and fall of the river from day to day. Measure the heights of the river on vertical lines, marking the lowest horizontal line low-water mark.

3. The latitude of a ship going east at noon on each of six consecutive days was as follows: - 1°, equator, + 1°, + 3°, + 1°, - 1°. Draw a graph showing the course of the ship.

4. The highest points reached by certain railroad stock during the 12 months of a year were as follows: 10 above par, 6 above par, 2 below par, 5 below par, par, 2 below par, 3 above par, 7 above par, 10 above par, 15 above par, 18 above par, 12 above par.

Draw a graph showing the variation in price of the stock during the year.

30. Addition of Positive and Negative Numbers. — The addition of positive and negative numbers may be indicated by writing them in succession with their signs.

Thus, to indicate the addition of + 6, + 2, - 4, + 3, and - 5, we write + 6 + 2 - 4 + 3 - 5. The first term being positive, we may omit its sign, giving 6 + 2 - 4 + 3 - 5.

Positive and negative numbers to be added may be placed — 3
also in columns, with the signs attached, as in the margin. 2

Since positive numbers are in nature ordinary arithmet- — 6
ical numbers, their sum is obtained by adding their absolute —
values, and is positive.

Thus, a bank deposit of \$125 and another of \$100 together make a de-
posit of \$225. The sum of + 4 and + 8 is + 12.

Also, to subtract each of two numbers in succession is equiva-
lent to subtracting their sum. Hence, two negative numbers,
since each is in nature a subtracted number, may be combined by
adding their absolute values, and the sum is negative.

Thus, a withdrawal from the bank of \$25 and an additional withdrawal of
\$75 together make a withdrawal of \$100. The sum of -6 and -10 is -16.

Therefore,

(1) *To add two numbers with like signs, find the sum of their
absolute values, and prefix the sign common.*

In adding two numbers with unlike signs, the one with the
greater absolute value may be separated into two parts, one of
which just neutralizes the number with the smaller absolute
value, leaving as result the other part of the numerically greater
number.

Thus, to add + 7 and - 3, by separating + 7 into + 3 and + 4, we have

$$\begin{array}{r} + 3 + 4 \\ - 3 \\ \hline 0 + 4 \end{array} \quad \text{or} \quad \begin{array}{r} + 7 \\ - 3 \\ \hline + 4 \end{array}$$

And, to add - 9 and + 4, by separating - 9 into - 4 and - 5, we have

$$\begin{array}{r} - 4 - 5 \\ + 4 \\ \hline 0 - 5 \end{array} \quad \text{or} \quad \begin{array}{r} - 9 \\ + 4 \\ \hline - 5 \end{array}$$

Hence, the following rule is evident:

(2) *To add two numbers with unlike signs, find the difference of
their absolute values, and prefix the sign of the one with the greater
absolute value.*

Evidently three or more positive and negative numbers may be added by adding two at a time, taking them in the order written. In the steps of the addition use rules (1) and (2).

A second method sometimes used when there are both positive and negative numbers to be added is to add the positive and the negative numbers separately, and then combine these two sums.

Thus, in this column, adding upwards, the partial sums are -2 , $+6$, and $+3$, the result. Or, the sum of -3 and -6 is -9 , and that of $+8$ and $+4$ is $+12$, and the sum of these sums is $+3$.

$$\begin{array}{r} -3 \\ +8 \\ -6 \\ +4 \\ \hline +3 \end{array}$$

NOTE.—If desirable, the numbers in the preceding illustration might be rewritten in the order shown in the margin. This fundamental principle that numbers to be added may be arranged in any order is known as the **Law of Order in Addition**.

$$\begin{array}{r} -3 \\ -6 \\ +8 \\ +4 \end{array} \left. \begin{array}{l} \\ \\ \\ \end{array} \right\}$$

The principle that numbers to be added may be grouped in any manner—by which we find the sum of the -3 and the -6 and of the $+8$ and $+4$ and then the sum of these sums, as here indicated—is known as the **Law of Grouping in Addition**.

EXERCISES

Find the sums of the following:

1. A profit of \$12 and a profit of \$16.
2. A loss of \$10 and a loss of \$23.
3. A profit of \$15 and a loss of \$6.
4. A profit of \$18 and a loss of \$30.
5. A rise in temperature of 14° and a rise of 6° .
6. A fall in temperature of 8° and a fall of 17° .
7. A rise in temperature of 7° and a fall of 16° .
8. A credit of \$250 and a debt of \$325.
9. A debt of \$136 and a debt of \$224.

$$\begin{array}{rcccccc} 10. & +6 & -8 & +9 & -12 & -20 & +8 \\ & \underline{-2} & \underline{-7} & \underline{+5} & \underline{+17} & \underline{-65} & \underline{-16} \end{array}$$

$$\begin{array}{r r r r r r}
 11. & -25 & -12 & +13 & -1 & +1 & -98 \\
 & +32 & -17 & +17 & +14 & -30 & +16 \\
 & \hline & \hline & \hline & \hline & \hline & \hline
 \end{array}$$

$$\begin{array}{r r r r r r}
 12. & -14 & +7 & -8 & +3 & -14 & +20 \\
 & +21 & -18 & +7 & -12 & -3 & +16 \\
 & \hline & \hline & \hline & \hline & \hline & \hline
 \end{array}$$

$$\begin{array}{r r r r r r}
 13. & -\$12 & +\$27 & -7 \text{ lb.} & +13 \text{ lb.} & +14 \text{ mi.} \\
 & -\$6 & -\$36 & +12 \text{ lb.} & +19 \text{ lb.} & -18 \text{ mi.} \\
 & \hline & \hline & \hline & \hline & \hline
 \end{array}$$

$$\begin{array}{r r r}
 14. & -8^\circ \text{ longitude} & +6^\circ \text{ latitude} & 25 \text{ ft. forward} \\
 & -17^\circ \text{ longitude} & -14^\circ \text{ latitude} & 36 \text{ ft. backward} \\
 & \hline & \hline & \hline
 \end{array}$$

$$\begin{array}{r r r r r r r}
 15. & -2 & 16. & +12 & 17. & -4 & 18. & +6 & 19. & -5 & 20. & +8 \\
 & +6 & & -7 & & -9 & & -3 & & +8 & & -16 \\
 & +3 & & -3 & & -3 & & -4 & & +14 & & -21 \\
 & -8 & & +6 & & -1 & & +12 & & -3 & & -7 \\
 & -4 & & +5 & & +12 & & +7 & & +6 & & -1 \\
 & \hline & \hline & \hline & \hline & \hline & \hline & \hline & \hline & \hline & \hline
 \end{array}$$

$$\begin{array}{r r r r}
 21. & -3.25 & -12.08 & +.62 & -4.125 \\
 & +4.09 & -10.37 & -6.07 & +1.450 \\
 & \hline & \hline & \hline & \hline
 \end{array}$$

$$\begin{array}{r r r r}
 22. & -3\frac{1}{2} & +6\frac{3}{8} & -\frac{1}{4} & +1\frac{5}{8} \\
 & +1\frac{1}{4} & -4\frac{1}{2} & -5\frac{2}{3} & +\frac{3}{4} \\
 & \hline & \hline & \hline & \hline
 \end{array}$$

$$23. \quad 5, -8, 7, -4, -12, 3.$$

$$24. \quad -9, -6, -2, 10, -8, 9, -1.$$

$$25. \quad 16, 5, 9, -5, 3, -7, -2, 6.$$

26. The Ahmes papyrus, the earliest mathematical book of which there is any record, was written about -1700 . The next great mathematical book was Euclid's *Elements*, written about 1400 years later. What was the date that Euclid wrote the *Elements*?

27. Euclid was born about -330 . Diophantus, who wrote one of the first books on algebra, was born 750 years later. At what date was Diophantus born?

28. A freight car is running at the rate of 20 feet per second. A brakeman walks to the rear on the top of the car at the rate of 5 feet per second. Express by $+$ and $-$ signs these rates and the rate and direction of his motion with reference to the ground.

29. If the basket of a balloon weighs 280 pounds, the instruments 57 pounds, the sandbags 800 pounds, and the balloon itself — 1400 pounds, what is the total weight of the balloon and contents?

30. A boat using both sails and steam is driven against a current by its steam power at the rate of 10 miles per hour, and by the wind at the rate of 3 miles per hour. The current flows at the rate of 4 miles per hour. Express these rates by $+$ and $-$ signs, and indicate their sum. What is the boat's rate of progress?

31. A work train travels 12 miles north, then 35 miles south, then 8 miles north. By use of $+$ and $-$ signs express these distances, their sum, and hence its final distance and direction from the starting point.

32. The temperature falls 15° , rises 18° , falls 10° , then rises 6° . By use of $+$ and $-$ signs express these changes and their net result.

33. The rise and fall in feet of the Mississippi River at Vicksburg on each of the seven days beginning Jan. 6, 1910, were given by the Weather Bureau as follows: $-.2$, $-.2$, $-.4$, $-.2$, $+.4$, $+.2$, $+.4$. The sign $-$ indicated fall. Find the net change during that period.

34. The *average* of two or more numbers is their sum divided by the number of them.

Find the average of the following temperatures: 8 A.M., -7° ; 9 A.M., -6° ; 10 A.M., -3° ; 11 A.M., -1° ; 12 Noon, $+1^{\circ}$; 1 P.M., $+4^{\circ}$; 2 P.M., $+7^{\circ}$; 3 P.M., $+8^{\circ}$; 4 P.M., $+8^{\circ}$; 5 P.M., $+5^{\circ}$; 6 P.M., $+3^{\circ}$; 7 P.M., $+1^{\circ}$.

35. At Madison, Wis., the minimum temperatures on the seven days beginning Jan. 4, 1910, were as follows: -12° ; $+10^{\circ}$; -14° ; -18° ; $+6^{\circ}$; -6° ; -10° . Find the average temperature for the week.

36. At Cheyenne, Wyo., the temperatures for six days, beginning Jan. 3, 1910, were as follows: -10° ; -2° ; 0° ; -2° ; $+14^{\circ}$; $+10^{\circ}$. Find the average temperature for the six days.

37. If one place is midway between two other places, its latitude or longitude equals one-half the sum of their latitudes or longitudes.

The latitude of Panama is approximately $+8^{\circ}$, and of Chicago $+42^{\circ}$. Find the latitude of Key West, Florida, which is approximately midway between these two.

38. The longitude of New Orleans is $+90^{\circ}$, and of Pekin, China, -116° . Berlin, Germany, is approximately midway between these two places. What is its longitude?

31. Subtraction of Positive and Negative Numbers.—To subtract a number (subtrahend), is to find a number (remainder) which added to it gives the minuend.

Thus, to subtract 6 from 9 is to find a number (3) which added to 6 gives 9.

Hence, the rule for subtracting positive and negative numbers may be obtained from addition.

(1) What number added to $+3$ gives $+5$?

$\begin{array}{r} +5 \\ +3 \\ \hline +2 \end{array}$	$\begin{array}{r} +5 \\ -3 \\ \hline +2 \end{array}$
--	--

Hence, by subtracting, $+3$ But, by adding, -3

Hence, to subtract $+3$ from $+5$ is equivalent to adding -3 to $+5$.

(2) What number added to $+3$ gives -5 ?

$\begin{array}{r} -5 \\ +3 \\ \hline -8 \end{array}$	$\begin{array}{r} -5 \\ -3 \\ \hline -8 \end{array}$
--	--

Hence, by subtracting, $+3$ But, by adding, -3

Hence, to subtract $+3$ from -5 is equivalent to adding -3 to -5 .

(3) What number added to -3 gives $+5$?

Show, as above, that to subtract -3 from $+5$ is equivalent to adding $+3$ to $+5$.

(4) What number added to -3 gives -5 ?

Show, as above, that to subtract -3 from -5 is equivalent to adding $+3$ to -5 .

From these examples it is evident that to subtract one number (subtrahend) from another (minuend) is equivalent to adding the subtrahend, *with its sign changed*, to the minuend. Therefore,

To subtract, first change the sign of the subtrahend, then proceed as in addition.

Thus, to subtract $\begin{array}{r} -4 \\ -7 \end{array}$ change to $\begin{array}{r} -4 \\ +7 \end{array}$ then add.

NOTE. — The pupil should learn not to rewrite the problem with the sign changed in the subtrahend, but to make the change *mentally*, and add at once.

EXERCISES

1. What number added to -6 gives -4 ?
2. What number added to 8 gives -2 ?
3. What number added to -4 gives 6 ?
4. What number added to 12 gives 7 ?
5. What number added to -15 gives -8 ?
6. What number added to -5 gives 0 ?
7. What added to $\$7$ loss will give $\$9$ gain?
8. What added to a withdrawal of $\$125$ from a bank will give a balance of $\$150$?
9. What added to a rise of 12° temperature will give a net fall of 18° ?
10. What added to a fall of 15° temperature will give a net fall of 8° ?

Subtract the following:

11.	$\begin{array}{r} +7 \\ -6 \end{array}$	$\begin{array}{r} +12 \\ -3 \end{array}$	$\begin{array}{r} +2 \\ -10 \end{array}$	$\begin{array}{r} +20 \\ -8 \end{array}$	$\begin{array}{r} +9 \\ -14 \end{array}$	$\begin{array}{r} +16 \\ -4 \end{array}$	$\begin{array}{r} 0 \\ -6 \end{array}$
-----	---	--	--	--	--	--	--

24. What is the difference in time between the year $+36$ and the year -40 ?

25. Augustus Cæsar lived from -60 to $+14$. How old was he when he died?

26. Thales, the founder of the first Greek school of mathematics and philosophy, was born in the year -640 and died in the year -550 . At what age did he die?

27. Pythagoras, who is believed to have given the first proof that in a right triangle the square on the hypotenuse equals the sum of the squares on the other two sides, was born in -569 and murdered in -500 . At what age was he murdered?

28. Euclid wrote his geometry, called the *Elements*, in about -300 . By $+1300$ the book had been carried into western Europe and translated there. How long was this after the book had been written?

29. What is the difference in longitude between Chicago and Berlin, the longitude of the former being approximately $+88^\circ$ and of the later approximately -13° ?

30. During the month of December, 1909, the maximum temperature at Chicago, as recorded by the Weather Bureau, was $+55^\circ$, and the minimum -7° . What was the amount of variation during the month?

32. Multiplication of Positive and Negative Numbers.— Multiplication as defined in arithmetic is a short form of addition, where the addends are all equal. For example, $3 \times \$25$ means $\$25 + \$25 + \$25$. This definition is only a special case of a more general definition which may be stated in the following way:

To multiply one number (multiplicand) by a second number (multiplier) is to use the first as we use 1 (unity) to obtain the second.

Thus, in multiplying 3 by 5, to obtain 5 by using 1, we take

$$1 + 1 + 1 + 1 + 1 = 5.$$

Hence, to obtain 5×3 , we take

$$3 + 3 + 3 + 3 + 3 = 15.$$

From this definition are obtained the rules for multiplying positive and negative numbers.

(1) To obtain the multiplier $+3$ we must add three 1's; *i.e.*

$$+3 = 1 + 1 + 1.$$

Hence, to obtain $(+3)(+5)$ we must add three $+5$'s.

That is, $(+3)(+5) = +5 + 5 + 5 = +15$, the product.

And to obtain $(+3)(-5)$ we must add three -5 's.

That is, $(+3)(-5) = -5 - 5 - 5 = -15$, the product.

(2) To obtain the multiplier -3 we must subtract three 1's from 0; *i.e.*

$$-3 = 0 - 1 - 1 - 1.$$

Hence, to obtain $(-3)(+5)$ we must subtract three $+5$'s from 0; *i.e.* change their signs and add them.

Hence, $(-3)(+5) = -5 - 5 - 5 = -15$, the product.

And to obtain $(-3)(-5)$ we must subtract three -5 's from 0; *i.e.* change their signs and add them.

Hence, $(-3)(-5) = +5 + 5 + 5 = +15$, the product.

From these examples the following principles are evident:

- (1) *The product of two numbers with like signs is positive.*
- (2) *The product of two numbers with unlike signs is negative.*
- (3) *The absolute value of the product of two numbers is the product of the absolute values of the numbers.*

In symbols:

$$(+a)(+b) = +ab.$$

$$(-a)(-b) = +ab.$$

$$(+a)(-b) = -ab.$$

$$(-a)(+b) = -ab.$$

EXERCISES

Give orally the products of the following:

1. $+3, +7.$

5. $-2, -10.$

9. $+5, -7.$

2. $-6, -2.$

6. $+7, +3.$

10. $+8, -3.$

3. $+4, +5.$

7. $-2, +5.$

11. $-2, -15.$

4. $-8, -3.$

8. $-3, +6.$

12. $-7, +4.$

- | | | |
|----------------|-----------------------------------|------------------------------------|
| 13. $+6, -8.$ | 19. $-5, -15.$ | 25. $-\frac{3}{5}, -\frac{4}{5}.$ |
| 14. $-5, -12.$ | 20. $+6, +20.$ | 26. $+1\frac{1}{2}, -\frac{2}{3}.$ |
| 15. $+12, +4.$ | 21. $-10, +14.$ | 27. $-2.5, -2.5.$ |
| 16. $-9, -8.$ | 22. $-\frac{1}{2}, -\frac{1}{2}.$ | 28. $+1.6, -1.6.$ |
| 17. $+12, -8.$ | 23. $+\frac{3}{4}, -\frac{2}{3}.$ | 29. $-.8, +.7.$ |
| 18. $-7, +12.$ | 24. $-\frac{5}{8}, +\frac{2}{5}.$ | 30. $-1.3, -1.1.$ |

31. If the lifting force of a balloon is $+$, the weight of the ballast is $-$. Adding weights is indicated by $+$, and removing weights by $-$. When 5 weights, each of 20 pounds, are thrown overboard, what is the effect on the upward pull of the balloon? Show that this is equivalent to saying that $(-5)(-20) = +100$.

32. When four weights, each of 15 pounds, are added to a balloon, what is the effect on the upward pull of the balloon? Show that this is equivalent to saying that $(+4)(-15) = -60$.

33. If 10 boxes of goods, each weighing 250 pounds, are removed from a freight boat, what is the effect of the downward pressure of the boat in the water? Show that this is equivalent to saying that $(-10)(+250) = -2500$.

33. The Product of More than Two Factors. — The product of more than two factors is obtained by performing one multiplication at a time, according to the principles in § 32.

Thus, to find $(-2)(-3)(+4)(-5)$, we have

$$\begin{aligned} (-2)(-3) &= +6 \\ (+6)(+4) &= +24 \\ (+24)(-5) &= -120, \text{ the product.} \end{aligned}$$

Show that the following rule holds for finding the product of more than two factors:

Find the product of the factors regardless of signs, and prefix $+$ or $-$ according as the number of negative factors is even or odd.

34. Powers of Positive and Negative Numbers. — If the factors are all equal, what is the product called?

The rule in § 33 will serve for finding powers of positive and negative numbers.

Thus, $(+5)^2 = (+5)(+5) = +25.$
 $(+3)^3 = (+3)(+3)(+3) = +27.$
 $(+2)^4 = (+2)(+2)(+2)(+2) = +16.$

Evidently, *any power of a positive number is positive.*

Again, $(-9)^2 = (-9)(-9) = +81.$
 $(-5)^3 = (-5)(-5)(-5) = -125.$
 $(-4)^4 = (-4)(-4)(-4)(-4) = +256.$
 $(-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32.$

Evidently, *any even power of a negative number is positive, and any odd power of a negative number is negative.*

EXERCISES

Find the products of the following:

- | | |
|---------------------|------------------------------|
| 1. $-1, 2, -3.$ | 7. $-1, -1, -2, -2.$ |
| 2. $-2, -4, -1.$ | 8. $2, -7, 4, -2, -1.$ |
| 3. $3, -2, -5, 2.$ | 9. $-1, -1, -1, -1, -1.$ |
| 4. $2, -5, -8, -1.$ | 10. $-4, 2, -1, -1, -1, -2.$ |
| 5. $-6, -2, -1, 3.$ | 11. $-1, -2, -3, 4, 1, -5.$ |
| 6. $4, -5, -2, 3.$ | 12. $2, 3, -1, -2, 1, -2.$ |

Find the values of:

- | | | |
|---------------|---------------|---------------------------|
| 13. $(+2)^4.$ | 19. $(-2)^5.$ | 25. $(-1)^3(-2)^3.$ |
| 14. $(-2)^3.$ | 20. $(-3)^3.$ | 26. $(-2)^2(-3)^2.$ |
| 15. $(-3)^2.$ | 21. $(+4)^4.$ | 27. $(-5)^2(+2)^3.$ |
| 16. $(+1)^6.$ | 22. $(-1)^9.$ | 28. $(-1)^6(-2)^3.$ |
| 17. $(-1)^5.$ | 23. $(-2)^8.$ | 29. $(-1)^5(-2)^3(-3)^2.$ |
| 18. $(-4)^3.$ | 24. $(-3)^5.$ | 30. $(+2)^3(-1)^4(-3)^3.$ |

If $a = 2$, $b = -3$, $c = -1$, find the value of:

- | | | |
|----------------|-------------------|-------------------------------|
| 31. b^3 . | 34. $2ab^2$. | 37. b^3c . |
| 32. a^2b^2 . | 35. $3ac^3$. | 38. $a^3 + b^3$. |
| 33. abc . | 36. $a^2b^2c^2$. | 39. $a^2 + b^2 + c^2 - abc$. |

35. Division of Positive and Negative Numbers. — Division is the inverse of multiplication. For example, to divide 24 by 8 is to find a number (3) by which 8 must be multiplied to give 24. Hence, the rules for division of positive and negative numbers come directly from the rules for multiplication.

Since $(+5)(+3) = +15$, then $(+15) \div (+3) = +5$.

Since $(+5)(-3) = -15$, then $(-15) \div (-3) = +5$.

Since $(-5)(+3) = -15$, then $(-15) \div (+3) = -5$.

Since $(-5)(-3) = +15$, then $(+15) \div (-3) = -5$.

Evidently, from these examples we get the same laws of signs in division as in multiplication:

(1) *If the dividend and divisor have like signs, the quotient is positive.*

(2) *If the dividend and divisor have unlike signs, the quotient is negative.*

EXERCISES

Divide:

- | | | |
|---------------------|-----------------------|--|
| 1. -12 by 3 . | 11. $+14$ by -14 . | 21. $+\frac{1}{2}$ by $-\frac{1}{2}$. |
| 2. -20 by -4 . | 12. -20 by -20 . | 22. $-\frac{1}{3}$ by $-\frac{2}{3}$. |
| 3. $+18$ by -3 . | 13. $+50$ by -5 . | 23. $-\frac{3}{4}$ by $+\frac{1}{2}$. |
| 4. -27 by $+9$. | 14. -100 by $+25$. | 24. $+\frac{1}{5}$ by $-\frac{2}{5}$. |
| 5. $+64$ by -16 . | 15. $+66$ by -11 . | 25. $+\frac{2}{3}$ by $+\frac{3}{4}$. |
| 6. -81 by -9 . | 16. -96 by -8 . | 26. $-\frac{3}{8}$ by $-\frac{1}{4}$. |
| 7. $+42$ by -7 . | 17. -82 by $+41$. | 27. $+6.4$ by $-.8$. |
| 8. -54 by -6 . | 18. $+63$ by -7 . | 28. -2.1 by $-.3$. |
| 9. $+72$ by $+8$. | 19. -84 by -12 . | 29. $-.63$ by $+.7$. |
| 10. -60 by -6 . | 20. $+132$ by $+11$. | 30. -1.44 by -1.2 . |

If $a = -2$, $b = 3$, and $c = -4$, find the value of:

$$\begin{array}{llll}
 31. \frac{a^2}{c} & 33. \frac{2ab^3}{c} & 35. \frac{a^5b^3}{c^2} & 37. \frac{abc}{6} \\
 32. \frac{b^2c^2}{a} & 34. \frac{3ac^2}{2b} & 36. \frac{c^3}{a^2b} & 38. \frac{a+b+c}{a+b}
 \end{array}$$

36. Positive and Negative Numbers in Equations.—In Chapter II we learned how to find the roots of equations by means of *axioms*. We may now consider the members of equations as made up of positive and negative terms.

Let us solve $8x - 7 = 3x + 3$.
 To remove -7 from the first member add $+7$ to both members.
 Then $8x = 3x + 3 + 7$.
 To remove $3x$ from the second member add $-3x$ to both members.
 Then $8x - 3x = 3 + 7$.
 Uniting terms, $5x = 10$.
 Dividing by 5, $x = 2$, the root.

Observe that to remove the *known term* from the first member and the *unknown term* from the second member, we proceed as follows:

(1) *The known term in the first member, with its sign changed, was added to both members.*

(2) *The unknown term in the second member, with its sign changed, was added to both members.*

NOTE.—The effect of adding $+7$ to both members in the equation above was to make the term -7 disappear from one member and reappear in the other, with its *sign changed*. Adding $-3x$ to both members had a similar effect. Evidently, adding such terms to both members of an equation is equivalent to *moving terms from one member to the other and changing the signs of the terms moved*.

This mechanical process of moving terms from one member to the other by changing their signs is called **transposition**. The terms are said to be **transposed**. Thus, by transposing $+2$ and $-2a$, $a + 2 = 5 - 2a$, becomes $a + 2a = 5 - 2$.

The process of transposition was discovered by the Arabs. About $+830$ an Arab named Mohammed ben Musa Al Khowarazmi wrote a mathematical book with the title *Aljabr Walmuqabalah*. The word *Aljabr* meant *restora-*

tion or transposition, as the term is now called. From this word came the name *algebra*.

The pupil should use the correct phraseology of "adding equals to both members of the equation" until the thing actually done is firmly fixed in mind.

37. Equations with Negative Roots. — The roots of some equations are negative numbers.

EXAMPLE 1. — Solve $7t + 15 = 4t + 3$.

Adding -15 and $-4t$ (transposing), $7t - 4t = 3 - 15$.

Uniting like terms, $3t = -12$.

Dividing by 3, $t = -4$.

EXAMPLE 2. — Solve $3V + 3 = 8 + 4V$.

Adding -3 and $-4V$ (transposing), $3V - 4V = 8 - 3$.

Uniting like terms, $-V = 5$.

Dividing by -1 , $V = -5$.

EXERCISES

Solve:

1. $9y = 7y - 28$.

2. $3A - 4 = 2A - 7$.

3. $5s + 20 = 4 - 3s$.

4. $3m + 27 = 6 - 4m$.

5. $5K - 15 = K - 35$.

6. $2t + 3 = 4t + 9$.

7. $7P - 8 = 11P + 12$.

8. $13 = 3x + 40$.

9. $6 - a = 12 + a$.

10. $17 = 19 + M$.

11. $B = 5 + 9B$.

12. $3 - g = 3g + 8$.

13. $W + 7 = 5W + 20$.

14. $\frac{Q}{5} - \frac{Q}{2} = 3$.

15. $8 - \frac{b}{4} = 10$.

16. $x = \frac{x}{2} - 9$.

17. $\frac{R}{2} + \frac{R}{3} = \frac{R}{6} - 4$.

18. $t + \frac{t}{2} + 10 = \frac{t}{4}$.

19. $6 + p + \frac{p}{3} = \frac{2}{3} - 4p$.

20. $5 + \frac{z}{3} - \frac{z}{2} = \frac{13}{2}$.

38. Interpretation of Negative Roots. — In solving a problem by means of an equation, a negative answer may have a *natural interpretation*, or it may indicate that the *problem is impossible*.

EXERCISES

Solve the following problems and interpret the answers:

1. On the second of three days the thermometer rose 20° more than on the first, and on the third day 13° more than on the second. The total rise during the three days was 35° . What was the change of temperature on the first day?

2. In a certain year a merchant, on the average, lost \$200 in each of the first 5 months, and made a profit of \$450 in each of the next 6 months. His net profit for the year was \$1550. Did he have a loss or profit in the last month, and how much?

3. A man gained \$1300 during three months. During the second month he gained \$650 more than the first month, and during the third month \$350 more than the second. Did he gain or lose during the first month, and how much?

4. A farmer sowed 18 acres of wheat more the second day than the first, 12 more the third than the second, and 5 more the fourth than the second. He sowed 59 acres in all. How many acres did he sow each day?

5. In a triangle the sum of the angles is 180° . The second is 12° less than one twelfth of the first, and the third 15° more than the first. How many degrees in each angle?

SUPPLEMENTARY EXERCISES

1. Herodotus, the Greek historian, commonly called the "Father of History," was born in -484 and died in -424. At what age did he die?

2. Livy, one of the most famous of the Roman historians, was born in -59 and lived to be 76 years old. In what year did he die?

3. A ship which leaves port at 42° north latitude goes one day 4° south, the next 3° north, the next 2° south, the next 3° south, and the next 1° south. If sailing south is called - and sailing north +, express the sum of these motions, and find the latitude of the ship at the end of the fifth day.

4. A switch engine, making up a freight train, moves forward 30 yards, back 120 yards, forward 150 yards, forward 40 yards, then back 300 yards. Represent these distances by $+$ and $-$ numbers, and find the position of the engine at the end of the work.

5. Find the average of the following temperatures: -10° , -6° , 2° , -4° , 3° , 7° , -4° , 5° , 16° , and -14° .

6. Add $-\frac{1}{2}$, $-\frac{1}{3}$, $+\frac{1}{4}$, $-\frac{1}{6}$, and $+\frac{1}{8}$.

7. Add $+14\frac{2}{3}$, $-8\frac{2}{3}$, $-11\frac{1}{6}$, and $-5\frac{4}{15}$.

8. Find the values of:

$$(-1)^7; (-2)^6; (-2)^4(-3)^3; (-5)(-4)^2(-3)^2.$$

9. If $A = -2$ and $B = -3$, find the value of $(A - B)^3$.

10. If $x = 4$ and $y = -2$, find the value of

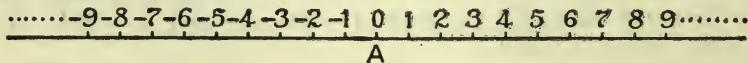
$$(x^2 + xy + y^2)(x^2 - xy + y^2).$$

11. Solve $\frac{3n}{2} - \frac{1}{3} = \frac{n}{6} - \frac{1}{2}$.

12. Solve $5 - \frac{4y}{5} = 8y + 9\frac{2}{5}$.

13. Positive and negative numbers may be added *graphically* as follows:

Let positive and negative numbers be represented by distances measured to the right and left, respectively, of a point A on a line, as in § 28.



To add a positive number count to the right as many spaces as there are units in the number, and to add a negative number count to the left as many spaces as there are units in the number.

Thus, to add -3 and $+7$, begin at -3 and count to the right 7 spaces. What is the result? To add -2 and -6 , begin at -2 and count to the left 6 spaces. What does it give?

Add graphically the following :

14. -5 and $+8$.

20. -2 , -3 , and $+4$.

15. $+6$ and -4 .

21. $+5$, $+2$, and -3 .

16. $+2$ and $+7$.

22. -6 , $+8$, and -4 .

17. -4 and -3 .

23. -1 , $+5$, -7 , and $+4$.

18. $+4$ and -9 .

24. $+4$, $+3$, -12 , and $+1$.

19. -7 and $+12$.

25. $+8$, -8 , -4 , and $+10$.

26. Show why a positive number is subtracted graphically by counting to the left the number of units in the subtrahend, and a negative number is subtracted by counting to the right the number of units in the subtrahend.

27. Subtract graphically the following: -3 from -7 ; $+2$ from -4 ; $+6$ from -8 ; -5 from $+5$; $+1$ from -1 ; -1 from $+1$; -6 from -6 ; $+8$ from -2 .

28. Viewing multiplication as a special form of repeated addition, where the addends are equal, show graphically that the product of two positive numbers is positive, and that the product of a negative number by a positive number is negative.

CHAPTER IV

ADDITION AND SUBTRACTION OF LITERAL EXPRESSIONS

39. Addition of Monomials. — In § 9 it was shown that similar terms are added or subtracted by adding or subtracting the coefficients, and attaching to the results the common letters with their exponents.

Thus, $3 \text{ lb.} + 4 \text{ lb.} = ?$ $3 l + 4 l = ?$ $6 \text{ sq. yd.} + 10 \text{ sq. yd.} = ?$
 $6 y^2 + 10 y^2 = ?$ $9 \text{ men} - 4 \text{ men} = ?$ $9 m - 4 m = ?$ $25 x^2 y - 14 x^2 y = ?$

Since positive and negative terms are by nature added and subtracted terms, respectively, they may be added by the same rule: that is,

To add two or more positive or negative terms, add their coefficients according to the rules in § 30, and attach to the result the common letters with their exponents.

For example, the sum of $-4 a^2 b$, $+7 a^2 b$, $+3 a^2 b$, and $-9 a^2 b$ is $-4 a^2 b + 7 a^2 b + 3 a^2 b - 9 a^2 b = -3 a^2 b$. The terms may be written in a column and added, as in the margin.

$-4 a^2 b$
$+7 a^2 b$
$+3 a^2 b$
$-9 a^2 b$
$-3 a^2 b$

EXERCISES

Add:

- | | |
|------------------------------------|---|
| 1. $5 a, 4 a, -2 a$. | 7. $-3 x^4, -x^4, 10 x^4, -5 x^4$. |
| 2. $9 n, -12 n, -3 n$. | 8. $PQ, -4 PQ, 16 PQ, -12 PQ$. |
| 3. $10 ab, 4 ab, -ab$. | 9. $-V^2, 9 V^2, 7 V^2, -8 V^2$. |
| 4. $6 P, -8 P, -10 P$. | 10. $b^2 c^3, -4 b^2 c^3, -b^2 c^3, -6 b^2 c^3$. |
| 5. $-7 m^3 n, 4 m^3 n, -5 m^3 n$. | 11. $5 B^2, -8 B^2, -12 B^2, 17 B^2$. |
| 6. $12 A, -4 A, -7 A, -9 A$. | 12. $-3 xy^5, -7 xy^5, xy^5, -2 xy^5$. |

$$\begin{array}{r r r r r}
 13. & -4a & 3x & -10K & 15y^3 & -20zw \\
 & 6a & -7x & -8K & -6y^3 & 13zw \\
 & \underline{-8a} & \underline{-9x} & \underline{12K} & \underline{y^3} & \underline{zw}
 \end{array}$$

$$\begin{array}{r r r r r}
 14. & 13xyz & -a^4 & -12pq & 100M & 35V^2 \\
 & -10xyz & 7a^4 & -10pq & -30M & -10V^2 \\
 & -6xyz & -4a^4 & 14pq & -4M & -25V^2 \\
 & \underline{-xyz} & \underline{-a^4} & \underline{-pq} & \underline{M} & \underline{-V^2}
 \end{array}$$

$$\begin{array}{r r r r r}
 15. & a^2bc & -s^3t^2 & 2ab^3 & -7T & -10N^6 \\
 & -4a^2bc & -4s^3t^2 & -3ab^3 & 9T & 4N^6 \\
 & 10a^2bc & 20s^3t^2 & 60ab^3 & -16T & 16N^6 \\
 & -15a^2bc & -8s^3t^2 & -32ab^3 & -T & -3N^6 \\
 & \underline{-a^2bc} & \underline{s^3t^2} & \underline{-15ab^3} & \underline{12T} & \underline{-14N^6}
 \end{array}$$

16. $-2.05A, 3.25A.$

20. $\frac{1}{2}y, -\frac{1}{4}y, -\frac{5}{8}y.$

17. $0.36x, -3.42x.$

21. $\frac{2}{3}a, -\frac{1}{6}a, -\frac{1}{2}a.$

18. $-1.2n, -3.4n, 7.1n.$

22. $3mn, -\frac{1}{2}mn, 2\frac{1}{2}mn.$

19. $-0.75P^3, -2.15P^3, P^3.$

23. $-2\frac{1}{3}v^3, -v^3, 3\frac{2}{3}v^3.$

Simplify:

24. $-3x + 7x + x - 6x.$

26. $-5at^2 - at^2 + 10at^2.$

25. $12c^4 + c^4 - 8c^4 - c^4.$

27. $6v' - 8v' - 10v' + v'.$

Simplify by adding similar terms:

28. $4a - 2a - 12x + 6a + 3x.$

SUGGESTION.— Adding the terms in a and the terms in x separately gives $8a - 9x.$

29. $x^2 - ab + 10ab - 6x^2 - 2ab.$

30. $3m^2 - 2mn - m^2 - mn.$

31. $a^2 + 2ab + b^2 - a^2 + 2ab - b^2.$

32. $y^3 - 3y^2z + 3yz^2 - z^3 - 2y^3 + z^2y - 4z^3.$

Add:

33. $4a$ and $-7b$.

SOLUTION. — Since these terms are not similar, their coefficients cannot be added. The addition may be merely *indicated*, giving $4a - 7b$.

Indicating the addition of dissimilar terms is called adding them.

34. $-2a$, $3b$, and $-c$.

35. h^2 , $-2hk$, and k^2 .

36. a^2b , $-ab^2$, a^3 , and $-b^3$.

NOTE. — The process of adding monomials by adding the coefficients may be used to shorten computations in arithmetical work. For example, $27 \times 54 + 73 \times 54 = 5400$, which is determined without the aid of a pencil.

At sight give the results of:

37. $6 \times 48 + 4 \times 48$.

43. $53 \times 84 - 13 \times 84$.

38. $8 \times 27 + 12 \times 27$.

44. $128 \times 96 - 28 \times 96$.

39. $17 \times 25 + 13 \times 25$.

45. $42 \times 28 + 16 \times 28 + 2 \times 28$.

40. $84 \times 63 + 16 \times 63$.

46. $34 \times 17 + 27 \times 17 - 11 \times 17$.

41. $46 \times 35 - 6 \times 35$.

47. $16 \times 38 - 4 \times 38 + 18 \times 38$.

42. $32 \times 46 + 18 \times 46$.

48. $16\pi + 48\pi + 36\pi$.

40. Addition of Polynomials. — It is evident that two or more polynomials are added if all of their terms are added. Hence, *two or more polynomials may be added by adding their similar terms.*

The process of adding polynomials is similar to that of adding denominate numbers involving two or more units of measure.

For example :

Just as	$\begin{array}{r} 12 \text{ bu. } 1 \text{ pk. } 2 \text{ qt.} \\ 5 \text{ bu. } 2 \text{ pk. } 5 \text{ qt.} \\ \hline 17 \text{ bu. } 3 \text{ pk. } 7 \text{ qt.} \end{array}$	so	$\begin{array}{r} 12b + p + 2q \\ 5b + 2p + 5q \\ \hline 17b + 3p + 7q \end{array}$
---------	---	----	---

To add polynomials, arrange the similar terms in columns and add each column as in § 39. The sums of the columns, connected by their signs, constitute the sum of the polynomials.

As pointed out in § 30 (See Law of Order in Addition, § 30, NOTE), numbers to be added may be arranged in any order without changing the value of their sum. In polynomials to be added, if the similar terms do not come in the same order in all the polynomials, they should be *rearranged* so that they do come in the same order.

Thus, to add $2a + 3b - 4c$, $5c - 4a - b$, and $b + 5a - 2c$, the terms in the last two expressions are rearranged, and the terms in a are put first, the terms in b second, and the terms in c third, as in the margin.

$$\begin{array}{r} 2a + 3b - 4c \\ - 4a - b + 5c \\ 5a + b - 2c \\ \hline 3a + 3b - c \end{array}$$

EXERCISES

Add:

$$\begin{array}{r} 1. \quad 3a - 2b \\ \quad 4a + 5b \\ \hline \end{array}$$

$$\begin{array}{r} 2. \quad -6x + y \\ \quad 4x - 7y \\ \hline \end{array}$$

$$\begin{array}{r} 3. \quad 4u - 2v + 3w \\ \quad -2u + v - 6w \\ \hline \end{array}$$

$$\begin{array}{r} 4. \quad -3H + K - 12L \\ \quad 8H - 4K + 3L \\ \quad -2H - 5K - L \\ \hline \end{array}$$

$$\begin{array}{r} 5. \quad -8m + 6n - p \\ \quad 5m - 2n + 3p \\ \quad -6m - 8n - 8p \\ \hline \end{array}$$

$$\begin{array}{r} 6. \quad 4r - 5r' - 2r'' \\ \quad -2r + 3r' - 7r'' \\ \quad -9r - r' + 4r'' \\ \hline \end{array}$$

$$\begin{array}{r} 7. \quad 3A - B + 2C \\ \quad 5A - 2B + C \\ \quad - A + 3B - 4C \\ \hline \end{array}$$

$$\begin{array}{r} 8. \quad 2P + 4Q + R - 7S \\ \quad -5P - Q + 3R + 2S \\ \quad 7P - 6Q - 9R + 3S \\ \hline \end{array}$$

$$\begin{array}{r} 9. \quad -20a + 3ab - 10b \\ \quad 16a - ab + 4b \\ \quad -7a - 5ab + 6b \\ \quad -2a - 8ab - 12b \\ \hline \end{array}$$

10. $2x - y$, $-3x + y$, and $4y - 7x$.

11. $5a + 3c - b$, $c - a + b$, and $2b - 3a + 4c$.

12. $33x^2 - 5xy + 2y^2$, $8xy - y^2 + 6x^2$, and $4y^2 - x^2 - xy$.

13. $1 - 2m + 3m^2$, $5m - 6 + 4m^2$, and $m^2 - 10m + 5$.

14. Show that the process of adding polynomials by adding the similar terms is used in adding numbers in arithmetic.

Thus, to add 723, 422, and 614, show that each number may be written as a polynomial, as in the margin. In what does this differ from the ordinary process of arithmetic?

$$\begin{array}{r} 723 = 7 \times 100 + 2 \times 10 + 3 \\ 422 = 4 \times 100 + 2 \times 10 + 2 \\ 614 = 6 \times 100 + 1 \times 10 + 4 \\ \hline 17 \times 100 + 5 \times 10 + 9 \\ \text{or } 1700 + 50 + 9 \\ \text{or } 1759 \end{array}$$

15. By writing them as polynomials, find by the method in Problem 14 the sums of the following:

831	123	302	915
614	502	824	201
<u>743</u>	<u>954</u>	<u>553</u>	<u>672</u>

41. **Checking Work.** — In much of the work of algebra one can easily *check* the results obtained; that is, determine whether or not mistakes have been made in the work.

A good method of checking addition of polynomials consists of assigning particular values to all of the general numbers involved in the polynomials and in the sum, and seeing if *the sum of the values* thus obtained for the polynomials is equal to *the value of the sum* of the polynomials.

EXAMPLE. — Add and check: $2a - 5b + 3c$, $3a + 4b - 2c$, and $5a - 2b - 4c$.

$$\left. \begin{array}{r} 2a - 5b + 3c = -5 \\ 3a + 4b - 2c = 9 \\ 5a - 2b - 4c = -3 \\ \hline 10a - 3b - 3c = 1 \end{array} \right\} \text{ when } a = 1, b = 2, c = 1.$$

When $a = 1$, $b = 2$, $c = 1$, the values of the polynomials are -5 , 9 , and -3 , respectively, and the sum of these values is 1 . The value of the sum of the polynomials also is 1 , which shows that the work is accurate. Any other values of a , b , and c might have been used. Check this work by giving a , b , and c other values.

NOTE. — This method may be used in checking much of the work in algebra that follows. While other methods of checking work may be practiced, it will be found that this method has the special advantage of helping the pupil to master more thoroughly the meanings of the symbols of algebra. *The pupil should form the habit of carefully checking all of his work in some way.*

EXERCISES

Add and check:

$$\begin{array}{r} 1. \quad 2x^2 - 3x + 1 \\ \quad 5x^2 + 7x - 4 \\ \hline \quad -3x^2 - 6x + 5 \end{array}$$

$$\begin{array}{r} 2. \quad a^3 - 3a^2 + 3a - 1 \\ \quad 4a^2 - a + 3 \\ \hline \quad -a^3 + 2a^2 + 2a \end{array}$$

$$\begin{array}{r} 3. \quad a^2 + 2ab + b^2 \\ \quad a^2 - 2ab + b^2 \\ \hline \quad 2a^2 \quad - 2b^2 \end{array}$$

$$\begin{array}{r} 4. \quad 26m + 10n + 14p \\ \quad -12m + 15n - 20p \\ \hline \quad -12n - 5p \end{array}$$

$$\begin{array}{r} 5. \quad 5u - 6v + 7w \\ \quad -3u - 2v + 9w \\ \hline \quad -8u - 5v + 11w \end{array}$$

$$\begin{array}{r} 6. \quad 5r - 6s + 8 \\ \quad -7r - 9s - 5 \\ \hline \quad 4r + 8s \end{array}$$

$$\begin{array}{r} 7. \quad x^3 \quad - y^3 \\ \quad x^3 + 6x^2y \\ \hline \quad - 6xy^2 - y^3 \end{array}$$

$$\begin{array}{r} 8. \quad -5an + 3bm + 4 \\ \quad -3an - 5bm - 1 \\ \hline \quad an + bm + 1 \\ \quad 7an - 2bm - 2 \end{array}$$

$$9. \quad 4s + 3t - 4, \quad 3s - 2t + 3, \quad -s + t + 5.$$

$$10. \quad 4n^2 - 3n + 7, \quad -2n^2 - 9n - 3, \quad 5n^2 + 4n - 9.$$

$$11. \quad 6a^2 + 5ab - 2b^2, \quad a^2 - ab - 8b^2, \quad 3a^2 + 7ab + b^2.$$

$$12. \quad 6x^3 - 4x^2 - 3x + 8, \quad 5x^2 + 3x + 2, \quad 8x^3 - 7x + 9, \quad x^2 - 2x.$$

$$13. \quad m^3 - n^3, \quad 2m^3 - 5mn^2 + 6n^3, \quad 3m^2n - 7n^3, \quad -m^3 - m^2n + mn^2 - n^3.$$

$$14. \quad A^6 - 2A^4 + 5A^2 - 3, \quad A^6 - 5A^4 - 6A^2, \quad 9A^4 + 7, \quad -6A^3 - A^2 - 9.$$

$$15. \quad v^2 - vt - t^2, \quad 3v^2 + 4vt - t^2, \quad t^2 - 3vt + v^2, \quad 7vt - 8t^2.$$

$$16. \quad 4ax - 7by - cz, \quad ax - 2by, \quad -3ax + 3cz, \quad 7by - 4cz.$$

$$17. \quad pq + 2qr - pr, \quad -2pq - 5qr + 4pr, \quad pq - pr.$$

$$18. \quad k - 2k^3 + 5k^2 - 3, \quad -3k^2 + 2k^3 + 6k - 3, \quad 2k^2 - 5, \quad 12 + 3k^2 - k.$$

$$19. \quad -12r + 15s - 8t, \quad 9r - 7s - 7t, \quad -9r + 4t, \quad -s + 5r, \quad 6r - 7t.$$

20. $3v' - 5v'' + v'''$, $3 - 4v'$, $-9v'' + v'''$, $7v' - v''$, $-6v'' + v'''$.
 21. $24P + W + 3H$, $5W - 9H$, $6P + 7H$, $-H - 5P$,
 $2H + 8W - P$.
 22. $5.6a - 2.7b$, $3.5a + 4.2b$, $-2.2a + b$, $a - 6.5b$.
 23. $.25x^2 - 1.3x + 8.1$, $-4.5x^2 - 1.9x - 4$, $6.3 - 5.8x + 3.5x^2$.
 24. $5R^2 - 2.5m$, $-.5R^2 + 4m$, $1.45R^2 + 3.2m$, $-.75R^2 - 9m$.
 25. $1 - 2D - 2.6D^2 + 7.5D^3$, $D^3 + 5.5D^2 - 7.2$, $-1.5 + 6.5D + 8D^2$.
 26. $\frac{1}{2}x - \frac{1}{3}y$, $\frac{1}{3}x + \frac{1}{2}y$, $x - y$.
 27. $\frac{1}{4}m - \frac{1}{2}n$, $2\frac{1}{2}m + \frac{1}{4}n$, $-\frac{1}{2}m + \frac{1}{4}n$.
 28. $\frac{2}{3}H + \frac{4}{5}K$, $1\frac{1}{3}H + \frac{1}{2}K$, $2H - 3K$.

42. Subtraction of Monomials. — What rule for subtracting positive and negative numbers was discovered in § 31? Since a monomial is itself a *number*, this rule holds equally in the subtraction of monomials.

To subtract one monomial from another, change the sign of the subtrahend and add the result to the minuend.

The change of sign of the subtrahend should be made *mentally*, the written sign being unchanged.

Thus, to subtract $9x^2y$ from $-18x^2y$, let the written work be as shown in the margin.

$$\begin{array}{r} -18x^2y \\ 9x^2y \\ \hline -27x^2y \end{array}$$

EXERCISES

Subtract:

- | | | | | | |
|----|---------|----------|---------|---------|----------|
| 1. | $2x$ | $-3a$ | $-6M^2$ | $8y^3$ | $-5ab$ |
| | $-3x$ | $-7a$ | $4M^2$ | $-2y^3$ | $-2ab$ |
| | <hr/> | <hr/> | <hr/> | <hr/> | <hr/> |
| 2. | $4m^2n$ | $-21xyz$ | $16A$ | $-6P$ | $-3ab^2$ |
| | $-m^2n$ | $-4xyz$ | $-14A$ | $-14P$ | $-ab^2$ |
| | <hr/> | <hr/> | <hr/> | <hr/> | <hr/> |
| 3. | $2t^3$ | $-7w^3v$ | $-hk$ | $6Q$ | $2abc$ |
| | $-3t^3$ | $2w^3v$ | $-4hk$ | $-22Q$ | $12abc$ |
| | <hr/> | <hr/> | <hr/> | <hr/> | <hr/> |

$$\begin{array}{r r r r r}
 4. \quad 3 w v^4 & 5 b y^6 & 21 x^2 y^2 z & 11 R^3 & - 4 v' \\
 - 6 w v^4 & - 8 b y^6 & - 3 x^2 y^2 z & - 3 R^3 & - 8 v' \\
 \hline
 \end{array}$$

5. From $2a$ take $-11a$.

8. From $-20P$ take $-9P$.

6. From $7x$ take $10x$.

9. From $32at^2$ take $-16at^2$.

7. From $-6t$ take $-4t$.

10. From $-16\pi r^2$ take $-7\pi r^2$.

11. $4x - (-3x) = ?$

13. $5m - (-2m) = ?$

12. $-3y - (-5y) = ?$

14. $16A - (-4A) = ?$

43. Subtraction of Polynomials.—It is evident that one polynomial is subtracted from another when all of its terms are subtracted. Consequently for subtraction of polynomials we have the following rule:

Add to the minuend the subtrahend with the sign before each of its terms changed.

Similar terms should be written in columns, as in addition.

The method of checking work used in § 41 may be used also for checking subtraction.

The change of signs should only be made *mentally*.

EXAMPLE 1.—From $3x^2 - 2xy + y^2$ take $5y^2 + 4xy + 3y^2$.

$$\left. \begin{array}{r}
 3x^2 - 2xy + y^2 = 2 \\
 5x^2 + 4xy - 3y^2 = 6 \\
 - 2x^2 - 6xy + 4y^2 = -4
 \end{array} \right\} \begin{array}{l} \text{when} \\ x = 1 \\ y = 1 \end{array}$$

If there is a term in the minuend and no term similar to it in the subtrahend, such a term will be written in a column by itself, and, in subtracting, it will be placed in the answer unchanged. If such a term occur in the subtrahend instead, in subtracting, it will be placed in the answer with its sign changed.

EXAMPLE 2.—From $4x^4 - 3x + 2x^5 + 7$ take $x^5 - 3x^3 + 2x^2 - x$.

$$\left. \begin{array}{r}
 2x^5 + 4x^4 \\
 x^5 - 3x^3 + 2x^2 - x \\
 x^5 + 4x^4 + 3x^3 - 2x^2 - 2x + 7 = 11
 \end{array} \right\} \begin{array}{l} \text{when} \\ x = 1 \end{array}$$

EXERCISES

Subtract and check:

$$\begin{array}{r} 1. \quad 4x - 5y \\ \quad x - 7y \\ \hline \end{array}$$

$$\begin{array}{r} 2. \quad C + 7D \\ \quad - 2C + 3D \\ \hline \end{array}$$

$$\begin{array}{r} 3. \quad 4T - 12h \\ \quad T - 6h \\ \hline \end{array}$$

$$\begin{array}{r} 4. \quad -m + 15a \\ \quad - 3m - 4a \\ \hline \end{array}$$

$$\begin{array}{r} 5. \quad 8b - 13d \\ \quad 9b + 8d \\ \hline \end{array}$$

$$\begin{array}{r} 6. \quad r - 2s + 5t \\ \quad r + 3s - 2t \\ \hline \end{array}$$

$$\begin{array}{r} 7. \quad 4x + y - 10z \\ \quad - x - 6y + 11z \\ \hline \end{array}$$

$$\begin{array}{r} 8. \quad -12p - 6q + 4r \\ \quad - 8p + q - 2r \\ \hline \end{array}$$

$$\begin{array}{r} 9. \quad u + v - w \\ \quad u - 2v + 3w \\ \hline \end{array}$$

$$\begin{array}{r} 10. \quad 3f - g + 2h \\ \quad 5f - 2g + h \\ \hline \end{array}$$

$$\begin{array}{r} 11. \quad 20p - q + r \\ \quad 2p + 5q - 7r \\ \hline \end{array}$$

$$\begin{array}{r} 12. \quad a^2 + 2ab + b^2 \\ \quad a^2 - 2ab + b^2 \\ \hline \end{array}$$

$$\begin{array}{r} 13. \quad x^3 + x^2 + x + 1 \\ \quad x^3 - x^2 + x - 1 \\ \hline \end{array}$$

$$\begin{array}{r} 14. \quad F^4 + 4F^2 + 1 \\ \quad 3F^4 - 2F^2 - 1 \\ \hline \end{array}$$

$$\begin{array}{r} 15. \quad 2ab - 3bc + 5ac \\ \quad 6ab + 7bc - 2ac \\ \hline \end{array}$$

$$\begin{array}{r} 16. \quad 2P + 4Q - 7R \\ \quad - 5P - Q + 3R \\ \hline \end{array}$$

$$\begin{array}{r} 17. \quad 3v - 4v' + 10v'' \\ \quad - 2v - 5v' + 12v'' \\ \hline \end{array}$$

$$\begin{array}{r} 18. \quad 7W - 5S + 9H \\ \quad 2W + 5S - 9H \\ \hline \end{array}$$

$$\begin{array}{r} 19. \quad a^6 - a^4 + a^2 - 1 \\ \quad 2a^6 - a^4 - a^2 - 1 \\ \hline \end{array}$$

$$\begin{array}{r} 20. \quad x^3 - 3x^2 + x + 5 \\ \quad 2x^2 - 5 \\ \hline \end{array}$$

$$\begin{array}{r} 21. \quad a^3 + 3a^2b + 3ab^2 + b^3 \\ \quad a^3 - a^2b + ab^2 - b^3 \\ \hline \end{array}$$

$$\begin{array}{r} 22. \quad 6 - 2m + 9m^2 - m^3 - m^4 \\ \quad 9 + m - 9m^2 + m^3 - m^4 \\ \hline \end{array}$$

$$\begin{array}{r} 23. \quad a^2 + 2ab + b^2 \\ \quad a^2 - b^2 \\ \hline \end{array}$$

$$\begin{array}{r} 24. \quad W^2 - V^4 \\ \quad - 6WV^2 - V^4 \\ \hline \end{array}$$

$$\begin{array}{r} 25. \quad 4 - t^2 + 7 t^4 \\ \quad - t^2 - 4 t^4 - 6 t^6 \\ \hline \end{array}$$

$$\begin{array}{r} 28. \quad Wt^2 - W^2t - 10 \\ \quad - W^2t + 15 \\ \hline \end{array}$$

$$\begin{array}{r} 26. \quad \textcirc{D} \quad 3n^2 - 2n + 1 \\ \quad n^3 - \quad n^2 \quad \textcirc{D} - 5 \\ \hline \end{array}$$

$$\begin{array}{r} 29. \quad 1.5a + 3.5b \\ \quad .7a - 1.3b \\ \hline \end{array}$$

$$\begin{array}{r} 27. \quad 8x - 2x' + 7x'' \\ \quad 9x + 6x' - 9x'' \\ \hline \end{array}$$

$$\begin{array}{r} 30. \quad \frac{1}{2}m - \frac{1}{4}n \\ \quad \frac{1}{4}m + \frac{1}{8}n \\ \hline \end{array}$$

31. From $6a - 7b - 10c$ take $12a + 4b - 7c$.

32. From $12x^2 - 3x + 1$ take $-2x^2 + 9x - 7$.

33. From $m^2 - 2mn + n^2$ take $4m^2 - mn + 3n^2$.

34. From $2 - 7A^2 + 4A$ take $3A - 5A^2 + 8$.

35. From $2t + 6t^2 + 1$ take $9t^2 + 4t + 3$.

36. From $15v' - 6 - 2v''$ take $-5v'' + 9 - v'$.

37. From $6a - 4a^3 + 7a^2$ take $10 - 3a + a^2 - 8a^3$.

38. From $6P - 5P^2 + 2P^3$ take $P^3 + 2P^2 - 7P + 3$.

39. From $D^2 + R^2 + 2DR$ take $5R^2 - 7DR$.

40. From $x^3 - 3x^2y + 3xy^2$ take $x^3 + x^2y - y^3$.

41. From $1.5r - s + 2.8t$ take $.9t - 3r - s$.

42. From $20E - 12.5E^2 + 4.2$ take $6 - 2.4E + 9E^3$.

43. From $5.6ab - 3k$ take $4.5k - 25$.

44. From $\frac{1}{3}x^2 - \frac{1}{2}x - \frac{1}{4}$ take $\frac{2}{3}x^2 - \frac{1}{3}x + \frac{3}{4}$.

45. $m^2 - 5m + 7 - (-2m^2 - 3m + 2) = ?$

46. $W^2 + 24W - 10 - (16 - 5W + W^2) = ?$

47. $a^2 - b^2 - (2ab - b^2) = ?$

48. $6 - (S - 4) = ?$

44. Removal of Signs of Grouping. — When the sign $-$ precedes a sign of grouping which incloses an expression, it indicates that the expression is a *subtracted* expression.

Subtraction is performed by changing the sign of each term of the subtrahend and adding the resulting expression to the minuend. Hence,

A sign of grouping preceded by the sign $-$ may be removed from an expression, if the sign before each term inclosed is changed.

For example, $3R - (5R - 2S) = 3R - 5R + 2S$.

And $-(-7y + 2y^2) = 7y - 2y^2$.

Since the sign $+$ expresses addition, or, when convenient, may be considered merely as a sign of distinction, therefore,

A sign of grouping preceded by the sign $+$ may be removed from an expression, without changing the sign of any of the terms inclosed.

For example, $4n + (3n - 7) = 4n + 3n - 7$.

And $+(8 - 2H) = 8 - 2H$.

When signs of grouping are inclosed within other signs of grouping, it is wise for the beginner to *remove the innermost sign first, then the innermost remaining sign, etc.*

Thus, $3a - (9a - \{2b - [6a - b]\})$
 $= 3a - (9a - \{2b - 6a + b\})$, removing brackets,
 $= 3a - (9a - 2b + 6a - b)$, removing braces,
 $= 3a - 9a + 2b - 6a + b$, removing parentheses,
 $= 3b - 12a$.

EXERCISES

Remove the signs of grouping and add similar terms:

1. $2r + (5r - 3)$.
2. $-8T - (7 - 2T)$.
3. $12 + \{-w + w^2\}$.
4. $C - [30C - 2]$.
5. $27 - (F - 32)$.
6. $a - (b - 2a - 5)$.
7. $2s + \overline{s + 10}$.
8. $5 - (4\pi - 1) - 2\pi$.
9. $6L + (9 - L) + 8$.
10. $2M + t - (5M - t)$.
11. $a^2 - (b^2 - 2ab)$.
12. $18y^2 - 7y + \{3y^2 - y\} - y$.
13. $6w - 4mn + (3mn - 2w)$.
14. $2B^2 - (B^2 - 4AC)$.
15. $(2r - 7h) - (r - 4h)$.

SUGGESTION.—Since neither of the signs of grouping incloses the other, the two may be removed at the same time, independently of each other.

16. $(1 - x + x^2) + (3x - 8)$.

17. $(2p - 5q) + (4p + q) + 6q$.

18. $W - (3w - 2.5) - (w - 1.5 W)$.

19. $P - \{3Q - 1\} + \{4Q - 6R\}$.

20. $x^2 + \{2x - (x^2 - 3x)\}$.

23. $-(-(-1))$.

21. $2v - [t - v + (3t - 4v)]$.

24. $-(-(-(-(-(-a))))))$.

22. $(2D - \{3 - D\} + 1)$.

25. $2t - (d - (2t - d))$.

26. $s - \frac{1}{2}at + \{2\frac{1}{2}at - (s - at)\} - \frac{1}{2}at$.

27. $m - (5n - (m - (2n - 6m)))$.

28. $a^2 - (2ab - (-(-b^2))) - (a^2 + (b^2 - 2ab))$.

29. Solve $6y - (4 - 2y) = 12$.

30. Solve $V + (3 + 5V) - 7 = 0$.

31. Solve $12.5k - (10 - 2k) = 19 - (k - 2)$.

32. Solve $3z - (8 - z) = z - (2z - 10)$.

33. Solve $u = 8 - (3u - 14 - \{6 - u\})$.

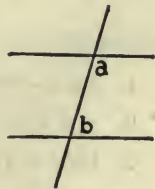
34. The sum of two numbers is 20 and their difference 6. What are the numbers?

SUGGESTION.—Let n = the larger number. Then $20 - n$ = the smaller.

35. The sum of two numbers is 7 and their difference 17. What are the numbers?

36. The sum of the angles a and b is 180° and their difference 30° . How many degrees in each?

37. My living expenses for two years amounted to \$2500. The expenses during the second year were \$280 more than during the first year. Find the expenses for each year.



45. Insertion of Signs of Grouping.—By reversing the processes in § 44, it follows that *terms of a polynomial may be inclosed*

within a sign of grouping, when this sign of grouping is preceded by the sign +, without changing the signs of the terms; and, when preceded by the sign -, by changing the signs of the terms.

Thus, to inclose the last three terms of $a^2 + 2bc - b^2 - c^2$ in parentheses preceded by the sign +, we have $a^2 + (2bc - b^2 - c^2)$. If preceded by the sign -, it becomes $a^2 - (b^2 - 2bc + c^2)$.

EXERCISES

Inclose the last three terms within a sign of grouping preceded by the sign + :

1. $A + B - C - D$.
2. $xy + yz - zw + wx$.
3. $5 - 2n - 9n^2 + n^3$.
4. $r - 4s + 4s^2 + 1$.
5. $m^2 + 6pq - q^2 - 9p^2$.
6. $x^2 + 2xy + y^2 - z^2 + 2zw - w^2$.
7. $16 - 8P + P^2 + 12M - 9M^2 - 4$.
8. $k + 7 + 10k - 1 - 25k^2$.
9. $18 - B^2 - b^2 - Bb$.
10. $1 + 6rr' - r^2 - 9r'^2$.

11-20. Inclose the last three terms in each of the above expressions within a sign of grouping preceded by the sign -.

21. In $an - 2t - 3n + bt$, place the terms involving n in parentheses preceded by the sign + and those involving t in parentheses preceded by -.

22. In $10P - 2W + mP - nW$, place the terms involving P in parentheses preceded by + and those involving W in parentheses preceded by -.

SUPPLEMENTARY EXERCISES

Simplify :

1. $5\sqrt{n} - 8\sqrt{n} + 6\sqrt{n} + \sqrt{n} - 10\sqrt{n}$.
2. $\sqrt{a+b} - 4\sqrt{a+b} + 6\sqrt{a+b} - 9\sqrt{a+b}$.
3. $12(x-y) - 3(x-y) - 16(x-y) + (x-y)$.
4. $-9(s-at^2) - 6(s-at^2) + (s-at^2) + 7(s-at^2)$.
5. $(2\pi R + 7)^2 + 14(2\pi R + 7)^2 - 8(2\pi R + 7)^2 + (2\pi R + 7)^2$.
6. $-20\sqrt[3]{8+\frac{1}{2}Wg^2} + 6\sqrt[3]{8+\frac{1}{2}Wg^2} + 18\sqrt[3]{8+\frac{1}{2}Wg^2} - 9\sqrt[3]{8+\frac{1}{2}Wg^2}$

Simplify by combining similar terms:

7. $16 - 2\sqrt{A} + 5\sqrt{A} - 24 - 7\sqrt{A}$.
8. $D + \sqrt{w} + \sqrt{t} - 4\sqrt{w} - 5D + 7\sqrt{w} - 6\sqrt{t}$.
9. $5(W - V) - 8W^2 + (W - V) + 17W^2 - 4(W - V)$.
10. $x^2 - 2xy + y^2 - 3x^2 - 6xy - 3y^2 + 4xy$.
11. $r_1^2 + 4r_1r_2 + r_2^2 - 2r_1^2 + 8r_1r_2 - 8r_2^2$.

Simplify by writing with a polynomial coefficient:

12. $ax + bx + cx$.

SOLUTION. — These terms are similar in x , and hence are added by adding the coefficients of x . Since the coefficients a , b , and c cannot be added into one term, but their sum can only be indicated, we get $(a + b + c)x$.

13. $my + ny - py$.
14. $aR^2 - bR^2 - cR^2 + dR^2$.
15. $Mt^3 + 2Nt^3 - 3Pt^3 + Qt^3$.
16. $5D^2 + \pi D^2 - hD^2 - 7D^2$.
17. $4a(x - y) - 3b(x - y) + 5(x - y)$.
18. Add $nA + mB + pC$ and $xA - yB + zC$.
19. From $aU - bV - cW$ subtract $gU + hV - kW$.

Remove signs of grouping and combine like terms:

20. $a + [2a - (3a - 2b)] - (4b - 3a)$.
21. $[m - (2m - n)] - [-3n + (4m - n)]$.
22. $(A - 5B) - \{3A - [4B + (6A - B) - (4B - 2A)]\}$.

In the following remove parentheses, leaving brackets, and simplify the results as far as possible:

23. $[(x - 2y) + (x + y)][(x - 2y) - (x + y)]$.
24. $[r - (s - t)][r + (s - t)]$.
25. $[5A + (3 + A)][5A - (3 + A)]$.
26. $[(4r - 3) + (3r' - 4)][(4r - 3) - (3r' - 4)]$.
27. $[(7a - 2b) - (c + 3d)][(7a - 2b) + (c + 3d)]$.

CHAPTER V

MULTIPLICATION AND DIVISION OF LITERAL EXPRESSIONS

46. Order of Factors.—Is there any difference between the values of 3×4 and 4×3 ? Of $2 \times 3 \times 4$ and $3 \times 4 \times 2$ or $4 \times 2 \times 3$ or $2 \times 4 \times 3$? Does it change the value of the product to change the order of the factors?

In general, has ab the same value as ba ? Has xyz the same value as xzy or yzx ? Has $2 \times a \times 3 \times b$ the same value as $2 \times 3 \times a \times b$?

Numbers to be multiplied may be taken in any order without changing the value of the product.

NOTE.—This fundamental principle is known as the **Law of Order in Multiplication**.

47. Grouping of Factors.—In $3 \times (2 \times 4)$ the parentheses indicate what? Is there any difference between the values of $3 \times 2 \times 4$ and $3 \times (2 \times 4)$? Of $2 \times 3 \times 4 \times 5$ and $(2 \times 3) \times (4 \times 5)$? Does it change the value of the product to group the factors so that the multiplications are performed in different ways?

In general, has abc the same value as $a(bc)$? Has $x^3x^2y^2y^4$ the same value as $(x^3x^2)(y^2y^4)$?

Numbers to be multiplied may be grouped in any way without changing the value of the product.

NOTE.—This fundamental principle is known as the **Law of Grouping in Multiplication**.

48. Law of Exponents in Multiplication.—How else is AA written? How else is $nnnn$ written? In x^3 what does the exponent show? $a^5 = aaaaa$. $a^6 = ?$ $R^7 = ?$

Multiplying a^3 by a^2 , we get

$$a^2 \times a^3 = aa \times aaa \text{ or } aaaaa = a^5.$$

Similarly,

$$m^4 \times m^2 = mmmmm = m^6.$$

$$v^2 \times v^3 \times v^5 = vvvvvvvvv = v^{10}.$$

$$D^4 \times D^5 = D^{4+5} = D^9.$$

In multiplying powers of the same base, the exponent of the product is obtained by adding the exponents of the factors.

In symbols, the above principle is expressed by

$$a^m \times a^n = a^{m+n}.$$

49. Multiplication of Monomials.—By use of the principles stated in the preceding sections the product of any two or more monomials may be found.

EXAMPLE 1.—To multiply $2x^2y$ by $3x^3y^3$, we have

$$\begin{aligned}(2x^2y)(3x^3y^3) &= (2 \times 3)(x^2x^3)(yy^3) \\ &= 6x^5y^4.\end{aligned}$$

EXAMPLE 2.—The product of $-5M^2N^5$ and $-4M^4N^2$ is

$$\begin{aligned}(-5M^2N^5)(-4M^4N^2) &= (-5)(-4)(M^2M^4)(N^5N^2) \\ &= 20M^6N^7.\end{aligned}$$

To multiply monomials, find the product of the numerical coefficients, using the laws of signs; then annex to the result the products of the like literal factors, using the law of exponents.

In performing multiplication, the work is often arranged in column form, as here shown.

$$\begin{array}{r} - 7v^2t^4 \\ 8v^7t \\ \hline - 56v^9t^5 \end{array}$$

EXERCISES

Find the product of:

- | | | |
|----------------------|-------------------------|--------------------------|
| 1. a^3 and a^4 . | 5. r^6 and r^5 . | 9. y and y^6 . |
| 2. x^2 and x^7 . | 6. V and V^4 . | 10. n^8 and n^{12} . |
| 3. b^3 and b^6 . | 7. L^9 and L^2 . | 11. W and W^8 . |
| 4. M^9 and M^2 . | 8. t^{10} and t^3 . | 12. T^2 and T^{11} . |

13. d^4 and d^9 . 16. z and z^{12} . 19. p^5 , p^2 , and p^2 .
 14. s^{12} and s^9 . 17. A^2 , A^3 , and A^4 . 20. x , x^2 , and x^4 .
 15. U^6 and U^8 . 18. C^3 , C^4 , and C^6 . 21. n , n^2 , n^8 , and n^5 .
 22. B , B^2 , B^6 , and B^9 . 29. c , $8c^2$, $3c^7$, and $2c^4$.
 23. $2x^4$, $3x^3$, and $2x^5$. 30. $-8d^2$, $-6d$, d , and $2d^8$.
 24. $4m$, $2m^4$, and m^7 . 31. x^7y^6 and x^6y^7 .
 25. $-2A$, $-3A^5$, and $4A^9$. 32. $-2a^3n^8$ and $5an^2$.
 26. $3R^3$, $-7R$, and $-2R^8$. 33. $-4AC$ and $3A^3B$.
 27. $-5y$, $-9y^3$, and $-2y^{10}$. 34. $7s$, $-2sr$, and $6s^3r^2$.
 28. B , $-6B^7$, and $4B^8$. 35. $-9x^2y^2z^2$ and $-4x^3z^4$.
 36. $3D^2W$, $-2D^2W$, and $8D^2W$.
 37. $-6M^3N^3$, $9M^2N$, and $-5N^3P$.
 38. $16x^4z^2$, $-2x^5z^2$, and $7y^3z^2$.
 39. $-2m^3n^5$, $8m^2p^4$, and $-5n^3p^2$.
 40. a^5 , $-2a^3b^2$, $7a^4c$, and $-3bc^3$.
 41. $\frac{1}{2}x^2y$, $-\frac{1}{2}x^3y^2$, and x^2y^3 .
 42. $-\frac{1}{3}HD$, $\frac{1}{2}H^2D$, and $3D^3$.
 43. $\frac{3}{4}rs^2$, $-2r^2s^2$, and $\frac{1}{2}r^3s^4$.
 44. $\frac{7}{8}pq^8$, $\frac{3}{7}p^6q^7$, and $2p^2q^5$.
 45. V , $\frac{1}{6}V$, $-4V^2$, and $\frac{1}{8}V^3$.
 46. $-\frac{3}{8}c^2d^{10}$, $\frac{4}{5}c^8d^2$, and $c^{10}d^6$.
 47. $\frac{1}{10}Wt^3$, $5W^2t^2$, and $4Wt^4$.
 48. $-MV^2$, $-\frac{1}{2}MV^2$, and $-6MV^2$.
 49. 3 , $-5Qt$, $8Q^2t$, and $-\frac{1}{4}Q^3t^2$.

50. Multiplication of a Polynomial by a Monomial. — The multiplication of a polynomial by a monomial is similar to the multiplication of denominate numbers containing two or more units of measure.

	9 yd.	2 ft.	6 in.		9 y + 2 f + 6 i
Just as			4	so	4
	36 yd.	8 ft.	24 in.		36 y + 8 f + 24 i

In general, since the whole of a quantity is multiplied by a number when each of its parts is multiplied by that number, we have the following rule:

To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial, and add the partial products obtained.

NOTE.—This fundamental principle, that a polynomial is multiplied by multiplying each of its terms separately and adding the partial products obtained, is known as the **Law of Distribution**. It is expressed in symbols by

$$n(a + b) = na + nb.$$

EXAMPLE. — The multiplication of $a^2 - 2ab + b^2$ by $3a^3b^3$ may be indicated either in the form

$$3a^3b^3(a^2 - 2ab + b^2) = 3a^5b^3 - 6a^4b^4 + 3a^3b^5$$

or as in the margin.

$$\begin{array}{r} a^2 - 2ab + b^2 \\ 3a^3b^3 \\ \hline 3a^5b^3 - 6a^4b^4 + 3a^3b^5 \end{array}$$

It is always more convenient in algebra to perform multiplications from left to right. This may also be done in arithmetic, but for convenience in "carrying" it is usually more convenient in arithmetic to begin at the right.

EXERCISES

Multiply :

1. $x + y$ by 5.
2. $a - b$ by 3.
3. $n - 4$ by 7.
4. $P + 1$ by 6
5. $2A - 3$ by 4.
6. $6a + 5b$ by 2.
7. $8M - 9N$ by 3.
8. $-5E + 7K$ by 10.
9. $4x - 9y$ by x .
10. $2a - 7b + c$ by 6.
11. $x^2 - 2x + 1$ by 4.
12. $a^2 + 2ab + b^2$ by ab .
13. $3 - 8m + 4m^2$ by $5m$.
14. $7a^2 + 12b^2$ by $-2a^2b$.
15. $15Rt - 4t^2$ by $4R^3$.
16. $4x^3 + 7y^3$ by $2x^2y^2$.
17. $3s^3 - 5s^2 + 9s$ by $4s^2$.
18. $B^2 - 4AC$ by $9AC$.
19. $5 - 2v + 7v^2$ by $-8v^3$.
20. $L^4 - 2L^2 + 3$ by $4L^2$.
21. $3n - 4 + 12p$ by $-2n^2p^3$.

22. $W^3 - 6W^2 + 2W - 3$ by $-5W^4$.

23. $2c^4 + 15 - 4c^3$ by $-3c^2$.

24. $A^2 + 2AB + B^2$ by $8A^3B^2$.

25. $ab + bc + ca$ by abc .

26. $2D^2 - 3DW + W^2$ by $-9D^2W^2$.

27. $A^3 - B^3$ by $2AB$.

28. $3x^4 - 6x^3 + x^2 + 8x - 1$ by $8x^6$.

29. $7ab + 3bc - 4cd$ by $2ac$.

33. $p^2(5p + 3q)$.

30. $5(4 - a - a^2)$.

34. $A(A^2 - 5AB + 6B^2)$.

31. $4x(3x + 8y)$.

35. $-2m^2(3 - 9m - 5m^2)$.

32. $7t(t^2 - at)$.

36. $3y^5(x^2 - 3xy + 7y^2)$.

37. $-8a^2b^2(-15a^3b + ab^3 - b^4)$.

38. $4m^3n^2(m^3 - m^2n + 3mn^2 - n^3)$.

39. $\frac{1}{2}(4s^2 - 6s + 2)$.

42. $-\frac{2}{3}x^3 + \frac{1}{2}x^2y - \frac{1}{3}xy^2 + \frac{1}{4}y^3$
 $-\frac{1}{3}x^2y^2$

40. $-\frac{1}{3}W(\frac{1}{4}W^2 - \frac{1}{3}WV + \frac{1}{6}V^2)$.

41. $\frac{1}{2}a^2 + \frac{1}{3}ab + \frac{1}{2}b^2$
 $\frac{1}{2}ab$

43. $1.2m^2 - 2.5mn + 3.4n^2$
 $-1.5m^3$

44. $2.15p^3 - .05p^2 + 1.2p - 4$
 $.5p^2$

Simplify by removing signs of grouping and uniting similar terms:

45. $2(2a - 3b) - 3a$.

46. $5(A - B) + 2(B - A)$.

47. $7(s^2 - 2st + t^2) + 3(2s^2 + st - 4t^2)$.

48. $a(ab - b^2) + b(a^2 + ab)$.

49. $3P(5 - 2P) + 4(3P + P^2)$.

50. $\frac{1}{2}S(2S^2 - \frac{1}{4}S) + \frac{1}{4}(2S^2 + S)$.

51. $4(2A - 3B) - 2(4A + B)$.

NOTE. — In this expression the term $-2(4A + B)$ may be looked upon in two ways, either as indicating that $4A + B$ is to be multiplied by 2 and the result subtracted from the part that precedes, or as indicating that $4A + B$ is to be multiplied by -2 and the result added algebraically to the part that precedes. In either case the terms within the parentheses ($4A + B$) will have their signs changed in the process.

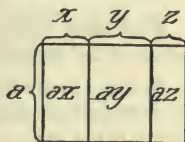
52. $6(c - 2k) - 4(2c - 3k)$. 54. $m(n^2 - mn) - n(mn + m^2)$.

53. $6P - 4Q - 3(7P + 2Q)$. 55. $-5(10r + 3t) - 2(2t - r)$.

56. $-3(x^2y - xy^2) - (xy^2 - x^2y)$.

57. $-3(v^3 + 2v^2w - 2w^3) - 2(-2v^3 + 3v^2w - w^3)$.

58. This rectangle is made up of three parts whose common width is a and whose lengths are x , y , and z , respectively. Show that the areas of the parts are ax , ay , and az , respectively. What does $a(x + y + z)$ represent? Hence, show that $a(x + y + z) = ax + ay + az$, which illustrates the *Law of Distribution*.



59. By a rectangle show in the same way that $2x(a + 2b + c) = 2ax + 4bx + 2cx$.

60. Show that the process of multiplying a polynomial by multiplying each of its terms and adding the partial products obtained is used in arithmetic in the multiplication of a number of two or more figures by a number of one figure.

Thus, to multiply 432 by 3, we see that 432 is in nature a polynomial, and may be written as the sum of three terms. 432 is multiplied by 3 by multiplying each of these terms by 3 and adding the results.

$$\begin{array}{r} 432 = 400 + 30 + 2 \\ 3 \qquad 3 \qquad 3 \qquad 3 \\ \hline 1296 = 1200 + 90 + 6 \end{array}$$

NOTE. — The ancients employed this principle in multiplication. Thus, by the Greeks the method of calculating 7×326 was equivalent (in modern notation) to that shown here. Because of the principle of *place value* in our notation, we perform our multiplications mechanically, without stopping to think of this Law of Distribution underlying the work.

$$\begin{array}{r} 7 \times 326 = 7(300 + 20 + 6) \\ = 2100 + 140 + 42 \\ = 2282 \end{array}$$

61. By the use of a rectangle as in Ex. 58, illustrate the *Law of Distribution* in finding 3×268

51. Multiplication of a Polynomial by a Polynomial. — The process of multiplying one polynomial by another is similar to the process in arithmetic of multiplying one number of two or more figures by another number of two or more figures. Just as in arithmetic the multiplicand is multiplied by the number represented by each figure of the multiplier, and all the partial products added, so in multiplication of polynomials in algebra the multiplicand is multiplied by each term of the multiplier, and all the partial products added.

For example:

	25		$2x + 5y$	
	34		$3x + 4y$	
Just as	$100 = 4 \times 25$	so	$6x^2 + 15xy$	$= 3x(2x + 5y)$
	$750 = 30 \times 25$		$8xy + 20y^2$	$= 4y(2x + 5y)$
	$850 = 34 \times 25$		$6x^2 + 23xy + 20y^2$	$= (3x + 4y)(2x + 5y)$

Each term of $2x + 5y$ is multiplied by $3x$ and then by $4y$, and the similar terms of the partial products are written in columns and added.

NOTE. — See that pupils clearly understand that we use this process abridged in arithmetical multiplication. Then have them see that algebraic multiplication is the same.

It is easily seen that this process is only an application of the *Law of Distribution* noted in § 50. Hence,

To obtain the product of two polynomials, multiply the multiplicand by each term of the multiplier, and add the resulting partial products.

The work in multiplication can be checked by assigning particular values to the general numbers involved, as in addition and subtraction, and seeing if the product of the particular values of the factors equals the particular value of the product.

EXAMPLE 1. — Multiply $2a - 3b + 5c$ by $3a + b - 2c$.

$$\begin{array}{rcl}
 2a - 3b + 5c & & = 4 \\
 3a + b - 2c & & = 2 \\
 \hline
 6a^2 - 9ab + 15ac & & \\
 2ab & - 3b^2 + 5bc & \\
 - 4ac & + 6bc - 10c^2 & \\
 \hline
 6a^2 - 7ab + 11ac - 3b^2 + 11bc - 10c^2 & = & 8
 \end{array}$$

When $a = 1$, $b = 1$, and $c = 1$, the multiplicand = 4, multiplier = 2, and product = 8, as it should.

Observe that the terms of the first partial product are written in a line by themselves, those of the second in a line by themselves, etc., and that the new kinds of terms, when obtained, are spaced out so that only similar terms will fall in columns.

NOTE. — A polynomial is said to be *arranged according to the powers of a letter* when the exponents of that letter either increase or decrease in the successive terms as we pass from left to right. Thus, $x^4 - 2x^3 + x^2 + x - 5$ is arranged according to the *descending* powers of x ; while $2y^4 + xy^3 + x^2y^2 + x^3y + x^6$ is arranged according to the *ascending* powers of x . How is the latter arranged as to y ?

It will be found an advantage in multiplication of polynomials to arrange, if possible, both the multiplicand and the multiplier alike according to the powers of some letter, before multiplying.

EXAMPLE 2. — Multiply $A^2 - 2 + 2A$ by $A - 6 + A^2$.

Arranging both trinomials according to the descending powers of A , we have

$$\left. \begin{array}{r} A^2 + 2A - 2 \\ A^2 + A - 6 \\ \hline A^4 + 2A^3 - 2A^2 \\ A^3 + 2A^2 - 2A \\ - 6A^2 - 12A + 12 \\ \hline A^4 + 3A^3 - 6A^2 - 14A + 12 = -4 \end{array} \right\} \begin{array}{l} = 1 \\ = -4 \\ \text{when} \\ A = 1 \end{array}$$

NOTE. — Substituting 1 for each of the letters in the two preceding examples tests the coefficients and signs only. Why? To test the exponents other values than 1 must be assigned to the letters.

EXERCISES

Multiply and test:

1. $n + 2$ by $n + 2$.
2. $A - 3$ by $A + 2$.
3. $x - 4$ by $x - 3$.
4. $2a + 3$ by $3a - 2$.
5. $5z + 6$ by $z + 4$.
6. $4t - 7$ by $4t + 3$.
7. $8b + 1$ by $3b + 1$.
8. $3R + 1$ by $4R + 1$.
9. $2V + 5$ by $3V + 7$.
10. $5y - 8$ by $6y - 1$.
11. $4D + 2$ by $-3D + 5$.
12. $2m + 6$ by $2m - 6$.

- | | |
|--|--|
| 13. $a - 10$ by $a + 10$. | 21. $2k - 3c$ by $2k - 3c$. |
| 14. $2W + 3$ by $2W + 3$. | 22. $x^2 - 2x + 1$ by $x + 2$. |
| 15. $3 + 4x$ by $2 + 3x$. | 23. $A^2 - 3A + 4$ by $A - 5$. |
| 16. $7 - S$ by $3 + S$. | 24. $v^2 + vw + w^2$ by $v + w$. |
| 17. $5 - 4b$ by $5 - 4b$. | 25. $M^2 - N^2$ by $M^2 + N^2$. |
| 18. $2a + 3b$ by $5a - 2b$. | 26. $2x - 3y + 1$ by $3x + 2y - 1$. |
| 19. $6p + 4t$ by $p - 2t$. | 27. $x^2 - 4 + 3x$ by $2 + 5x$. |
| 20. $5B + C$ by $5B + C$. | 28. $6 - 5R$ by $4R + 7$. |
| 29. $a^3 - a^2 + 4a - 6$ by $2a^2 - 3a + 2$. | |
| 30. $3n^2 - n + 4$ by $4n - 3$. | |
| 31. $1 - 2S + 3S^2$ by $4S + 5$. | |
| 32. $2 + 3T + T^2$ by $2 + 3T + T^2$. | |
| 33. $4a + 6b + 10c$ by $2a - 3b + 4c$. | |
| 34. $R - 5S + 3W$ by $3R + 2W - S$. | |
| 35. $\frac{1}{2}a + \frac{1}{3}$ by $\frac{1}{2}a + \frac{1}{3}$. | |
| 36. $2n + 6p - 5r$ by $2p - 3n$. | |
| 37. $a - 1$ by $1 + 2a + a^2$. | |
| 38. $3b^3 - 5b^2 + b - 4$ by $3 - 2b$. | |
| 39. $D - 2D^2 + 4$ by $8 + 2D - D^2$. | |
| 40. $2w^3 - 5w^2 + 4w - 1$ by $3 - 2w^2 - w$. | |
| 41. $(x + 5)(2x - 5)$. | 46. $(3 - v)(v^2 + 6 - v)$. |
| 42. $(2A + 4)(5A - 1)$. | 47. $(a^3 + b^3)(a^3 - b^3)$. |
| 43. $(1 - 4t)(1 + 4t)$. | 48. $(n^2 - 4)(n^4 + 6n^2 + 5)$. |
| 44. $(B^2 - 4AC)(B^2 + 4AC)$. | 49. $(a - 3t)(2a^2 + at - t^2)$. |
| 45. $(x^2 - 2x + 1)(x - 1)$. | 50. $(\frac{1}{4}x + \frac{1}{5}y)(\frac{1}{4}x - \frac{1}{5}y)$. |

Find the product of:

51. $x - 1$, $x + 2$, and $x + 1$.

SUGGESTION.—Multiply $x - 1$ by $x + 2$, and that product by $x + 1$.

52. $a + 3, a - 2, 2a + 1.$ 56. $2 + d, 2 - d, 4 + d^2.$
 53. $2B - 1, B + 2, 3B - 2.$ 57. $2p - 3t, 3p + 2t, t + p.$
 54. $3V - 4, 3V - 4, 1 + V.$ 58. $S^2 + S + 1, S - 1, S^3 + 1.$
 55. $m + n, m - n, m^2 + n^2.$ 59. $x^3 - z^3, x^3 + z^3, x^6 + z^6.$
 60. $(A + 1)(A + 2)(A + 3)(A + 4).$

Remove signs of grouping and simplify by combining terms:

61. $(t + 2)(t + 3) + (2t - 1)(t - 4).$
 62. $(2 + F)(3F - 1) + F(5F - 2).$
 63. $b^2 - 2b + (1 - b)(1 + b).$
 64. $(x + w)(x + w) + (x - w)(x - w).$
 65. $(a + 2)(a + 5) - (2a + 1)(a - 2).$

SUGGESTION.—The product of $(2a + 1)(a - 2)$ is to be subtracted from $(a + 2)(a + 5)$. What operation must be performed first?

66. $(A + B)(A + B) - (A - B)(A - B).$
 67. $2(y^2 - 3y + 1) - (y + 4)(y - 1).$
 68. $(v + t)(v - 2t) - (v - t)(v + 2t).$
 69. $n^2 - 3n + 4 - (2 - n)(3n + 2).$
 70. $(2a + 3b)(2a + 3b) - (2a - 3b)(2a - 3b).$

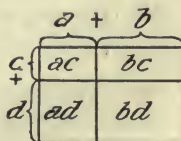
71. Show by the accompanying diagram that $(a + b)(c + d) = ac + bc + ad + bd$.

72. Show by a diagram, as in Problem 71, that $(x + 5)(x + 7) = x^2 + 12x + 35$.

73. Show by a diagram, as in Problem 71, what the product of $(a + b + c)(x + y + z)$ is.

74. Show how the *Law of Distribution* is used in multiplying the following:

$$\begin{array}{r} 234 \\ \underline{42} \end{array} \qquad \begin{array}{r} 506 \\ \underline{63} \end{array} \qquad \begin{array}{r} 427 \\ \underline{135} \end{array}$$



75. A rectangular field is 16 rods longer than it is wide. A second field is 5 rods shorter and 2 rods wider than the first. If w represents the width of the first, express the area of the second.

76. A rectangle is 8 inches longer than it is wide. If its length is increased by 4 inches, and its width decreased by 2 inches, represent by use of one letter its resulting area.

52. Law of Exponents in Division. — Since division is the inverse of multiplication, the law of exponents in division is obtained from the law in multiplication.

Since $a^3 \times a^2 = a^5$, $a^5 \div a^3 = a^2$.

Since $n^4 \times n^6 = n^{10}$, $n^{10} \div n^4 = n^6$.

Tell how the exponent of each quotient is obtained from the exponents of the dividend and divisor.

By use of cancellation we arrive at the same conclusion.

For example,
$$\frac{x^7}{x^3} = \frac{\cancel{x} \cancel{x} \cancel{x} \cancel{x} \cancel{x} x x}{\cancel{x} \cancel{x} \cancel{x}} = xxx = x^4.$$

In dividing powers of the same base, the exponent of the quotient is obtained by subtracting the exponent of the divisor from the exponent of the dividend.

In symbols this rule is expressed by

$$a^m \div a^n = a^{m-n}.$$

53. Division of Monomials. — By reversing the rule for multiplying monomials the rule for division is obtained.

Since $2a \times 3b = 6ab$, $6ab \div 2a = 3b$.

Since $(7x^3)(-3x^5) = -21x^8$, $(-21x^8) \div (7x^3) = -3x^5$.

In dividing monomials, divide the coefficient of the dividend by that of the divisor, using the laws of signs; then divide the literal factors by subtracting the exponents of the letters in the divisor from the exponents of the like letters in the dividend.

Thus,
$$(6x^4y^6) \div (-2xy^4) = -3x^3y^2.$$

Evidently, if a letter has the same exponent in the dividend and divisor, this letter will not appear in the quotient. Why?

Thus, $(-24x^3y^6) \div (-4x^3y^4) = 6y^2.$

Here x does not appear in the quotient.

This division may be indicated also in the form

$$\frac{-24x^3y^6}{-4x^3y^4} = 6y^2, \quad \text{or} \quad \begin{array}{r} 6y^2 \\ -4x^3y^4 \overline{) -24x^3y^6} \end{array}$$

EXERCISES

Divide:

- | | | |
|--------------------------------------|--|----------------------------------|
| 1. a^4 by a^3 . | 9. s^9 by s^8 . | 17. a^3b^3 by a^2b^2 . |
| 2. N^6 by N^4 . | 10. P^4 by P^3 . | 18. x^4y^2 by x^2y^2 . |
| 3. R^3 by R^2 . | 11. x^{18} by x^9 . | 19. m^8n^3 by m^4n^2 . |
| 4. y^7 by y^3 . | 12. A^{11} by A^7 . | 20. $a^2b^4c^6$ by $a^2b^2c^2$. |
| 5. t^8 by t^2 . | 13. m^{12} by m^{10} . | 21. $6x^6$ by $2x^2$. |
| 6. K^{10} by K^6 . | 14. D^4 by D^4 . | 22. $8Rv^3$ by $2Rv^2$. |
| 7. v^5 by v^3 . | 15. b^7 by b^5 . | 23. $4\pi R^2$ by $2\pi R$. |
| 8. x^6 by x^5 . | 16. z^5 by z . | 24. $18Wg^2$ by $9W$. |
| 25. $9a^8$ by $-3a^3$. | 33. $-50x^3d^4$ by $2x^2$. | |
| 26. $15t^7$ by $-5t^3$. | 34. $\frac{1}{2}A^4$ by $\frac{1}{2}A^3$. | |
| 27. $-24P^4n$ by $-6P$. | 35. $6z^7t$ by $4z^4t$. | |
| 28. $-56M^5N^3$ by $7N^2$. | 36. $-2m^{12}n$ by $3m^8n$. | |
| 29. $36a^5b^3c^2$ by $-9a^2b^3c^2$. | 37. $\frac{1}{3}Qt^3$ by $2Qt^2$. | |
| 30. $4x^{10}y^8$ by $-x^6y^7$. | 38. $\frac{1}{4}x^6y^6z^6$ by $-\frac{1}{2}x^4y^4$. | |
| 31. $-6Vt^2$ by $2Vt^2$. | 39. $7F^2n^4$ by $-\frac{1}{2}n^4$. | |
| 32. $18D^4v$ by $6D^2$. | 40. $-xD^5$ by $-\frac{1}{8}xD^3$. | |

54. Division of a Polynomial by a Monomial. — The division of a polynomial by a monomial is similar to the division of denominate numbers containing two or more units of measure.

Just as
$$\begin{array}{r} 5 \text{ ft. } 3 \text{ in.} \\ 3 \overline{) 15 \text{ ft. } 9 \text{ in.}} \end{array} \quad \text{so} \quad \begin{array}{r} 5f + 3i \\ 3 \overline{) 15f + 9i} \end{array}$$

In general, since *divisor* $\times$ *quotient* = *dividend*, the quotient must be such that the product of its terms by the divisor will give the terms of the dividend. Hence the rule:

To divide a polynomial by a monomial, divide each term of the dividend by the divisor, and add the partial quotients.

EXAMPLE. — Divide $4x^3y - 8x^2y^2 + 6xy^3$ by $-2xy$.

We divide each term of the dividend by $-2xy$ and set the results down in a line with their signs, which indicates their sum. The work may be written in either the form

$$(4x^3y - 8x^2y^2 + 6xy^3) \div (-2xy) = -2x^2 + 4xy - 3y^2$$

$$\text{or} \quad \begin{array}{r} -2x^2 + 4xy - 3y^2 \\ -2xy \overline{) 4x^3y - 8x^2y^2 + 6xy^3} \end{array}$$

EXERCISES

Divide:

1. $4m - 6n$ by 2.
2. $12D + 9F$ by 3.
3. $16x + 24$ by 4.
4. $21a - 28b$ by 7.
5. $36 + 42y$ by 6.
6. $-24P + 72Q$ by -12 .
7. $81ab - 18ac$ by $9a$.
8. $x + 2x^2$ by x .
9. $A^3 - 3A^2$ by A^2 .
10. $30m^3 + 40m^2n$ by $5m^2$.
11. $2pq - 4pt + 6pv$ by $2p$.
12. $-6W^4g + 18W^3g^2 - 12W^2g^3$ by $6W^2g$.
13. $x^6 - x^3$ by x^3 .
14. $a^4 - a^3b + a^2b^2$ by a^2 .
15. $27t^5 - 18t^3$ by $-9t^3$.
16. $-x - y + z$ by -1 .
17. $-A + 2B - C$ by -1 .
18. $4w^4v^2 + 8w^3v^3 - 12w^2v^4$ by $4w^2v^2$.
19. $12p^6t - 8p^4t^2 + 24p^2$ by $4p^2$.
20. $-21x^8 - 35x^6 - 14x^4$ by $-7x^4$.
21. $(8ab - 4bc - 20bd) \div 4b$.
22. $(z^2xy - 2zyx^2 + 3zy^3) \div zy$.
23. $(2P^2t^5 - 3P^3t^2 + P^4t^3) \div P^2t^2$.

$$24. (-a^3 - ta^2 + t^2a) \div (-a).$$

$$25. (M^2N^4 - 9MN^3) \div (-MN^3).$$

$$26. (a^2K - 7b^2K + 2cK) \div (-K).$$

$$27. (\frac{1}{2}n^2 + \frac{1}{4}n) \div \frac{1}{2}n.$$

$$28. (-\frac{3}{8}xy - \frac{1}{4}x^2y) \div (-\frac{1}{8}xy).$$

$$29. (\frac{1}{3}Wg - \frac{1}{6}Wg^2) \div (\frac{1}{3}Wg).$$

$$30. (.5t + t^2 - 1.5t^3) \div (.5t).$$

55. Division of One Polynomial by Another.—Let us divide $x^3 + x^4 + 7x^2 - 6x + 8$ by $x^2 + 2x + 8$.

First, arrange both dividend and divisor according to the descending powers of x (see § 51). The work may be arranged as below.

$$\begin{array}{r}
 x^2 - x + 1 \\
 x^2 + 2x + 8 \overline{) x^4 + x^3 + 7x^2 - 6x + 8} \\
 \underline{x^4 + 2x^3 + 8x^2} \\
 - x^3 - x^2 - 6x + 8 \\
 \underline{- x^3 - 2x^2 - 8x} \\
 x^2 + 2x + 8 \\
 \underline{x^2 + 2x + 8} \\
 0
 \end{array}$$

EXPLANATION.—The product of the term of highest power in x in the quotient and the term of highest power in the divisor must give the term of highest power in the dividend. Hence, the highest term of the quotient is obtained by dividing the highest term of the dividend x^4 by the highest term of the divisor x^2 . This gives x^2 , the first term of the quotient.

Multiply the whole divisor by the term of the quotient just found. This gives $x^4 + 2x^3 + 8x^2$, which is placed below the dividend.

The dividend is the product of the divisor by the whole quotient. Hence, subtracting the product $x^4 + 2x^3 + 8x^2$ from the dividend, the remainder $-x^3 - x^2 - 6x + 8$ must be the product of the divisor by the part of the quotient to be found.

Therefore, the product of the next highest term of the quotient by the highest term of the divisor must equal the highest term of the remainder. Hence, dividing $-x^3$ of the remainder by x^2 of the divisor gives $-x$, the second term of the quotient.

Multiply the whole divisor by the new term, $-x$; subtract the product from the remainder. This leaves $x^2 + 2x + 8$.

Evidently the third term of the quotient will be obtained from this second remainder just as the second term was obtained from the first remainder.

By continuing this process, all of the terms of the quotient may be found.

If in any problem the divisor is an *exact divisor* of the dividend, the work may be carried on until a remainder zero is found. Otherwise the work may be continued until a remainder is obtained in which the highest term is of lower power than the highest term of the divisor. This is a *true remainder*.

Hence the rule for dividing one polynomial by another:

1. *Arrange both the dividend and divisor according to the descending or ascending powers of some letter.*
2. *Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.*
3. *Multiply the whole divisor by this term of the quotient, and subtract the result from the dividend.*
4. *Treat the remainder as a new dividend (having the terms arranged as before), and repeat the process, continuing until either the remainder zero, or a true remainder, is found.*

EXAMPLE 1. — Divide $a^2 - 11a + 30$ by $a - 5$.

$$\begin{array}{r}
 a - 6 \\
 a - 5 \overline{) a^2 - 11a + 30} \\
 \underline{a^2 - 5a} \\
 - 6a + 30 \\
 \underline{- 6a + 30} \\
 0
 \end{array}$$

CHECK. — When $a = 1$, dividend = 20, divisor = -4 , and quotient = -5 , as it should.

EXAMPLE 2. — Divide $x^4 + y^4$ by $x + y$.

$$\begin{array}{r}
 x^3 - x^2y + xy^2 - y^3 \\
 x + y \overline{) x^4} \\
 \underline{x^4 + x^3y} \\
 - x^3y \\
 \underline{- x^3y - x^2y^2} \\
 x^2y^2 \\
 \underline{x^2y^2 + xy^3} \\
 - xy^3 + y^4 \\
 \underline{- xy^3 - y^4} \\
 2y^4
 \end{array}$$

In this example there is a true remainder $2y^4$ obtained. By using the fractional form to indicate the division of the remainder, as in arithmetic, the entire quotient in this problem may be expressed thus:

$$\frac{x^4 + y^4}{x + y} = x^3 - x^2y + xy^2 - y^3 + \frac{2y^4}{x + y}.$$

CHECK. — When $x = 2$ and $y = 1$, dividend = 17, divisor = 3, quotient = $5\frac{2}{3}$, as it should.

NOTE. — Observe that the terms in the dividend, divisor, and all the products and remainders in the above example are arranged in descending powers of x . Similar terms should be kept in columns in the work. For this purpose it is often necessary to leave vacant spaces between the terms, as in this example.

EXERCISES

Divide and check:

1. $a^2 + 3a + 2$ by $a + 1$.
2. $N^2 + 3N - 40$ by $N + 8$.
3. $x^2 - 5x + 6$ by $x - 2$.
4. $P^2 + 7P + 12$ by $P + 3$.
5. $t^2 - 6t + 8$ by $t - 4$.
6. $y^2 - 9y + 20$ by $y - 4$.
7. $H^2 - H - 30$ by $H - 8$.
8. $b^2 + b - 20$ by $b + 5$.
9. $m^2 + 3m - 40$ by $m - 10$.
10. $2x^2 + 3x + 1$ by $2x + 1$.
11. $3x^2 + 16x - 12$ by $x + 6$.
12. $12K^2 - K - 6$ by $4K - 6$.
13. $6V^2 + 29V + 35$ by $3V + 7$.
14. $4A^2 - 9$ by $2A + 3$.
15. $25R^2 - 4$ by $5R - 2$.
16. $4p^2 - 1$ by $2p + 1$.
17. $1 - 9a^2$ by $1 - 3a$.
18. $16 - 25D^2$ by $4 + 5D$.
19. $a^2 - b^2$ by $a - b$.
20. $x^2 - 4y^2$ by $x + 2y$.
21. $W^2 - 25g^2$ by $W - 5g$.
22. $m^2 + 5mn + 4n^2$ by $m + 4n$.
23. $P^2 - 9PQ + 20Q^2$ by $P - 5Q$.
24. $x^2 + 21 + 10x$ by $3 + x$.
25. $1 + 2a + a^2$ by $a + 1$.
26. $15y + 36 + y^2$ by $y + 4$.
27. $12E^3 - 30E^2 + 2E - 5$ by $2E - 5$.
28. $2r^3 - 3r^2 + 3 + 4r$ by $3 + 2r$.
29. $B^3 - 1$ by $B - 1$.
30. $y^3 - z^3$ by $y + z$.
31. $W^4 - V^4$ by $W^2 + V^2$.
32. $H^6 - t^6$ by $H^3 + t^3$.
33. $144R^2 - 1$ by $12R + 1$.

34. $b^3 + 3b^2 + 3b + 1$ by $b^2 + 2b + 1$.
35. $x^3 + x^4 + 1$ by $x^4 - x^2 + 1$.
36. $n^6 - a^6$ by $n^2 - a^2$.
37. $2y^4 - 9y^3 - 14y + 17y^2$ by $y^2 - 2y$.
38. $8d^3 + 27t^3$ by $2d - 3t$.
39. $4q^4 - 9q^2 + 20q - 25$ by $2q^2 + 3q - 5$.
40. $A^3 - 2A^2 + 5$ by $A - 5$.
41. $g^4 - 3g^2k + 6k^2$ by $g^2 - 3k$.
42. $a^4 + n^4 + a^2n^2$ by $n^2 + a^2 - an$.
43. $16h^4 - 96h^3 + 81 - 216h + 216h^2$ by $9 + 4h^2 - 12h$.
44. A rectangular field whose area is $x^3 + 3x^2y + 3xy^2 + y^3$ has a length of $x^2 + 2xy + y^2$. What is its width?
45. Show that the problem $682 \div 31$ may be written in the form $(6t^2 + 8t + 2) \div (3t + 1)$. Find the quotient in this form, and compare the answer with that obtained by the arithmetical process.
46. In the problem $882 \div 42$ express the given numbers as polynomials and find the quotient. Compare the answer found with that obtained by the arithmetical process.
47. Divide $1 + x$ by $1 - x$. How many terms are there in the quotient?
48. Divide $1 + 2n$ by $1 + n$, carrying the quotient to four terms.
49. Divide $1 + y$ by $1 - y$, carrying the quotient to four terms.

SUPPLEMENTARY EXERCISES

Find the product of:

- | | |
|---|--|
| 1. a^3 , $-2a^2$, and $-3a$. | 6. $a^n b^{n+1}$ and $a^{n+1} b^n$. |
| 2. x^n and x^{2n} . | 7. $-8W^2v$ and $3W^d v^d$. |
| 3. y^m , y^{3m} , and y^{4m} . | 8. $5z^{a-5}$, $-2z^{a+7}$, and z^{a-3} . |
| 4. $3t^a$, $-4t^{a+1}$, and $-2t^{a-1}$. | 9. $a^n \cdot a^n \cdot a^n \cdot a^n \cdot a^n$. |
| 5. $-5R^{a+b}$, $-2R^{a-b}$, and R^{3a} . | 10. $x^m \cdot x^m \cdot x^m \cdot x^m \dots$ to n factors |

Divide:

- | | |
|----------------------------|--|
| 11. w^{17} by w^6 . | 15. $12 x^{4n}$ by $-6 x^n$. |
| 12. M^a by M^b . | 16. $-63 A^{3n}$ by $-7 A^5$. |
| 13. y^{5m} by y^{2n} . | 17. $48 x^{a+4} b^{a-4}$ by $-6 x^4 b^{a-4}$. |
| 14. B^{a+5} by B^a . | 18. $-80 v^{mt^n}$ by $16 v^{m-5} t^{n-1}$. |

Multiply:

- | | |
|---|---|
| 19. $a^n + b^n + c^n$ by a^n . | 22. $P^n + Q^n$ by $P + Q^n$. |
| 20. $x^{n-3} - y^n + z^{n+3}$ by $x^3 y^3$. | 23. $a^x + b^y$ by $a^x - b^y$. |
| 21. $x^a - y^b$ by $x^a - y^b$. | 24. $x^n - y^n$, $x^n + y^n$, and $x^{2n} + y^{2n}$. |
| 25. $V^i + 1$, $V^i - 1$, $V^{2i} + 1$, $V^{4i} + 1$. | |

Divide:

26. $a^{2v} - a^{3v}$ by a^v .
27. $10 a^{n+2} - 4 a^{n+3}$ by $2 a^2$.
28. $M^c + M^{c+2} + M^{c+4}$ by M^c .
29. $y^{n+1} x^{2n+1} + y^{2n+1} x^{n+1}$ by $x^{n+1} y^{n+1}$.
30. $x^6 - y^6$ by $x^2 - y^2$.
31. $W^{4n} - V^{4n}$ by $W^{2n} - V^{2n}$.
32. $a^{3m} + b^{3m}$ by $a^m + b^m$.
33. $S^{4a} - T^{4a}$ by $S^a - T^a$.
34. $x^3 + 3 x^2 y + 3 x y^2 + y^3 + z^3$ by $x + y + z$.

SUGGESTION. — Arrange the dividend, divisor, all partial products, and remainders according to the descending powers of x , without reference to the other letters involved. If two or more terms in any case contain the same power of x , it does not matter which precedes.

35. $a^3 + b^3 - c^3 + 3 abc$ by $a + b - c$.
36. $m^3 + n^3 + p^3 - 3 mnp$ by $m + n + p$.
37. $u^2 + v^2 + w^2 + 2 uv + 2 uw + 2 vw$ by $u + v + w$.
38. $4 a^2 + 4 ab + b^2 - 12 ac - 6 bc + 9 c^2$ by $2 a + b - 3 c$.

CHAPTER VI

LINEAR EQUATIONS: PROBLEMS

56. In Chapters II and III we found how to solve problems by means of equations containing one unknown number. With a knowledge of the principles discovered in Chapters IV and V at command, we are now able to consider additional points connected with the solutions of problems, including the solutions of problems by means of equations containing more than one unknown number.

57. **Degree of an Equation.** — The degree of a term is the sum of the exponents of its literal factors. If there is only one literal factor, its exponent is the degree of the term.

For example :

x is of the *first degree*,
 $5x^2$ is of the *second degree*,
 $2a^2b$ is of the *third degree*,
 $3w^2v^2$ is of the *fourth degree*.

The degree is expressed sometimes with respect to some one letter.

Thus, $4An^2t^3$ is of the first degree with respect to A , of the second degree with respect to n , and of the third degree with respect to t .

The **degree of an equation** is the degree of its term of highest degree with respect to the unknown number or numbers.

Thus,

$n + 3 = 4n$ is of the *first degree*,
 $x^2 - 3x = 8y$ is of the *second degree*,
 $R^3 + R^2 - 6 = 0$ is of the *third degree*,
 $y^2z^2 - yz = 2y + 3$ is of the *fourth degree*.

An equation of the first degree is called a **linear equation**.

One of the second degree is called a **quadratic equation**.

One of the third degree is called a **cubic equation**.

58. Linear Equations with One Unknown Number. — The process of solving a linear equation with one unknown number that does not contain a sign of grouping was given in § 15 and § 36. In solving linear equations that contain signs of grouping, the latter should first be removed by use of the principles in Chapters IV and V.

EXAMPLE 1. — Solve $P - (6 - 2P) = 9(P - 1)$.

Here the first parentheses are removed by changing the signs of the terms inclosed, and the second by performing the indicated multiplication, giving :

$$\begin{aligned} P - 6 + 2P &= 9P - 9, \\ P + 2P - 9P &= 6 - 9, \\ -6P &= -3, \\ P &= \frac{1}{2}. \end{aligned}$$

CHECK. — When $P = \frac{1}{2}$ the equation becomes

$$\begin{aligned} \frac{1}{2} - (6 - 1) &= 9(\frac{1}{2} - 1), \\ -4\frac{1}{2} &= -4\frac{1}{2}. \end{aligned}$$

OR

EXAMPLE 2. — Solve $(n - 2)(n - 3) = (n - 4)(n - 5)$.

The parentheses are removed by multiplication, giving :

$$\begin{aligned} n^2 - 5n + 6 &= n^2 - 9n + 20, \\ n^2 - n^2 - 5n + 9n &= 20 - 6, \\ 4n &= 14, \\ n &= 3\frac{1}{2}. \end{aligned}$$

CHECK. — When $n = 3\frac{1}{2}$ the equation becomes

$$\begin{aligned} (3\frac{1}{2} - 2)(3\frac{1}{2} - 3) &= (3\frac{1}{2} - 4)(3\frac{1}{2} - 5), \\ \frac{3}{4} &= \frac{3}{4}. \end{aligned}$$

OR

EXERCISES

Solve and check :

1. $3(a + 1) = 12 + 4(a - 1)$.
2. $3(x - 2) = 2(x - 3)$.
3. $5(2 - 4y) = 4(1 - 3y)$.
4. $5(4 - 3R) = 7(3 - 4R)$.
5. $2t - (5t + 5) = 7$.
6. $3(v + 1) + 5(v - 1) = 0$.
7. $3(d - 14) = 7(d - 18)$.
8. $2(W - 1) + 3(W - 2) + 4(W - 3) = 0$.
9. $2P - 5(20 - P) - 6 = 0$.
10. $5(2D + 1) - 7 = 3(2D - 7) + 51$.

11. $2(m-2) + 3(m-3) = 20 - 4(m-4)$.
12. $5k + 6(k+1) = 7(k+2) + 8(k+3)$.
13. $2c - 3(c-4) + 20 = 2c + 17$.
14. $85 - 3(2V+7) = 6V + 4(4V+2)$.
15. $4(5+4E) - 5(6+4E) + 100 = 2E + 36$.
16. $12T - (4T-7) = 3T - (9T-28)$.
17. $(5-n)(1+n) = (2-n)(4+n)$.
18. $(K-1)(K-2) = (K+3)(K-4)$.
19. $(s-1)(s-4) = 2s + (s-2)(s-3)$.
20. $(5Y+7)(3Y-8) = (5Y+4)(3Y-5)$.
21. $(4A-7)(4A-7) = (2A-5)(8A+3)$.
22. $(5-3u)(4u+3) + 1 = (3u+7)(1-4u)$.
23. $(Q+4)(Q+8) - Q(Q+4) = 128$.
24. $(4-3w)(5+4w) = (8+2w)(1-6w) - 82$.
25. $(a-4)(a+4) = a^2 - 8a$.
26. $2(z+2)(z-4) = z(2z+1) - 21$.

59. Problems solved by Linear Equations.—The following example illustrates a type of problems solved by means of linear equations involving signs of grouping.

EXAMPLE.—A man made two investments in railroad stocks, amounting together to \$15,000. On the first he gained 14%, and on the second he lost 5%. His net profits from the two investments amounted to \$1055. How many dollars in each investment?

Let a = amount of first investment, in dollars.

Then, $15000 - a$ = amount of second investment.

$.14a$ = gain on first investment.

$.05(15000 - a)$ = loss on second investment.

Hence, $.14a - .05(15000 - a) = 1055$.

$.14a - 750 + .05a = 1055$.

$.19a = 1805$.

$a = 9500$.

$15,000 - a = 5500$.

Therefore the first investment was \$9500, and the second, \$5500.

For the steps in the process of expressing a problem by means of an equation involving one unknown letter, as illustrated in the above example, review § 16.

EXERCISES

1. The sum of two numbers is 25, and twice the larger exceeds four times the smaller by 2. Find the numbers.
2. Separate 34 into two parts such that twice the greater shall be less by 12 than 5 times the smaller.
3. The sum of two numbers is 64. Three times the less is 12 more than twice the greater. Find the numbers.
4. Separate 86 into two parts such that three times the greater shall exceed 5 times the smaller by 2.
5. There are two consecutive odd numbers whose product exceeds the square of the less by 34. Find the numbers.
6. When the square of a whole number is increased by 29 the result is found equal to the product of the two next larger consecutive whole numbers. Find the three consecutive numbers.
7. I bought 4 lb. of coffee and 5 lb. of tea for \$5.40. The tea cost 45 cents a pound more than the coffee. Find the cost of each per pound.
8. A newsboy sold 51 papers for 70 cents, of which some were one-cent papers and the others two-cent papers. How many papers of each kind were there?
9. A newsboy sold two-cent evening papers and *Saturday Evening Posts* (price 5 cents), receiving in all 92 cents. The total number of papers and *Posts* together was 34. How many were there of each?
10. Some boys had a refreshment booth at which they sold lemonade and cider. The price of lemonade was 2 cents a glass and of cider 5 cents a glass. In an afternoon they sold 43 glasses, and in counting their money found that they had \$1.31. How many glasses of each did they sell?

11. Two tanks contained equal amounts of oil. But after 75 gallons had been taken from one, and 50 gallons added to the other, one contained twice as much as the other. How many gallons did each contain originally?

12. What amount must be subtracted from each of the numbers 12, 14, 18, 10, so that the product of the first two remainders shall equal the product of the last two?

13. Two electric lights have together a strength of 128 candle-power. One has twice as much and 16 candle-power more than the other. How many candle-power has each?

14. Two grades of coffee costing the dealer 28 cents and 36 cents per pound are to be mixed so that the mixture shall cost 30 cents per pound. What parts must he take to make 50 pounds of the mixture?

15. How can a merchant mix 10 pounds of tea, one kind costing 50 cents and the other 65 cents per pound, so that the mixture shall cost 60 cents per pound?

16. Two grades of spice worth 25 cents and 45 cents a pound are to be mixed so that the mixture can be sold for 50 cents a pound and at a profit of 25 %. What parts must be taken to make 8 pounds of the mixture?

17. A grocer received a shipment of 80 dozen eggs. Part of these were sold at 28 cents per dozen and the rest at 30 cents per dozen. The total receipts from the sale of the eggs amounted to \$23.12. How many dozens were sold at each price?

18. Listerine contains 25 % alcohol. When used to spray the throat, it should be diluted by adding water to it. How much water must be added to 100 parts of listerine so that the mixture contains only 15 % alcohol?

19. If a medicine contains 40 % alcohol, how much of other ingredients must be added to 10 quarts of it so that the mixture shall contain only 25 % alcohol?

20. How many quarts of water must be mixed with 40 quarts of alcohol, 80 % pure, to make a mixture 75 % pure?

21. How many quarts of water must be added to 60 quarts of alcohol, 85 % pure, to make a mixture 75 % pure?

22. How many gallons of cream containing 28 % butter fat and milk containing 4 % butter fat must be mixed to make 10 gallons of cream containing 25 % butter fat?

23. How many gallons of cream containing 30 % butter fat and milk containing 4.5 % butter fat must be mixed to make 20 gallons of cream containing 20 % butter fat?

24. In an alloy of gold and silver weighing 60 ounces, there are 5 ounces of gold. How much silver must be added in order that 10 ounces of the new alloy shall contain only $\frac{10}{17}$ ounce of gold?

25. In an alloy of copper and tin weighing 36 ounces, there are 12 ounces of copper. How many ounces of copper must be added to this in order that the new alloy may have 4 ounces of copper to every 10 ounces of the alloy?

26. A man has two investments together amounting to \$1800. On the first investment he gets 5 % annually, and on the other he gets 6 % annually. His annual income from the two investments is \$95. Find the amount of each investment.

27. \$5000 is invested in two places, part at 4 % and the rest at 5 %. The annual income from the two investments is \$222. How much is each investment?

28. One sum of money exceeds another by \$1400. The first at 8 % and the second at 6 % give an annual income of \$280. Find the value of each.

29. There are 51 coins in a money drawer, consisting of nickels and dimes. The total value is \$3.35. How many coins of each kind are there?

30. The value of 29 coins, consisting of quarters and dimes, is \$4.55. Find the number of each.

31. A is 42 years of age, and B is 12. In how many years will A be only twice as old as B?

32. Eight years ago a man was just 16 times as old as his son, and now he is only 4 times as old. What are their present ages?

33. Find where to cut a board which is 50 inches long into two parts whose difference is 26 inches.

34. A bar of iron 60 inches long is to be cut into two parts such that twice one part is equal to 5 times the other. Find where to cut it.

35. A belt runs over a pulley 48 inches in diameter, making 180 revolutions a minute. It is desired to reduce the size of the pulley so that by making 216 revolutions a minute, the belt will move with the same speed. By how much must the diameter of the pulley be reduced?

36. A rectangular field is 6 rods longer than it is wide; and if the length and breadth were each 4 rods more, the area would be 120 square rods more than it is. What are the dimensions of the field?

37. A square court has the same area as a rectangular court whose length is 18 feet greater and width 9 feet less than the side of the square court. Find the side of the square court.

38. A tennis court is 6 feet more than 2 times as long as it is wide, and its area exceeds by 216 square feet twice the area of the square whose side is its width. Find the dimensions of the court.

60. **Business Problems.** — Certain kinds of business problems are easily solved by use of linear equations.

EXERCISES

1. A university has an endowment which invested at 4% yields an annual income of \$140,000. What is the endowment?

SUGGESTION. — If d dollars is the endowment, $.04d = 140,000$.

2. What sum of money invested at 6% will yield \$510 simple interest per year?

3. A certain investment at $4\frac{1}{2}\%$ simple interest yields \$3240 in 6 years. What is the investment?

4. A certain sum of money was invested at 5 % simple interest. In 3 years the principal and interest together amounted to \$6900. What was the amount of the investment ?

5. Five years ago I invested a certain sum of money at 6 % simple interest. It now amounts to \$1560. How much did I invest ?

6. A number of years ago I invested \$1500 at 5 % simple interest. The amount at present is \$2025. How many years ago was the investment made ?

7. A commission merchant charged \$10.80 for selling a car load of fruit, and remitted \$349.20. What was his rate of commission ?

8. Merchants sometimes mark goods to sell at an advance, and then allow a discount on the marked price. At what advance must a merchant mark goods costing \$12 in order that he may sell them at a reduction of 20 % from the marked price, and yet make a profit of 25 % on the goods ?

SUGGESTION. — If a = the advance, in dollars, the marked price will be $12 + a$, and the selling price 80 % of this, or $.80(12 + a)$. Hence, $.80(12 + a) = 12 + .25 \times 12$.

9. At what advance must a merchant mark an article costing him \$3.50 in order that he may sell it at a reduction of 10 % from the marked price, and yet make a profit of 30 % on it ?

10. A clothing merchant puts on sale a lot of boys' suits costing him \$8 each, and advertises them to sell at a reduction of 25 % from the marked price. At what advance must he first mark them to sell in order that he may make this reduction and still make a profit of 20 % on the cost of the suits ?

11. A milliner advertises a lot of hats that cost her \$2.85 each for special sale at " $\frac{1}{3}$ off." At what advance must she first mark them in order that by allowing this discount she may make a profit of 75 cents on each hat ?

12. The proprietor of a china store marks for sale a lot of dishes that cost him \$16.50. At what advance must he mark them so that he may then mark them down 15 % and make a profit of 20 % ?

13. A ladies' tailoring establishment marked a suit at an advance of 50 %, then reduced the selling price by 20 % of the marked price, and got \$16.80 for it. What was the cost of the suit?

14. A druggist had marked perfume to sell at a profit of 40 %. He then advertised it for special sale at 42 cents an ounce, which was a reduction of $14\frac{2}{7}$ % from the marked price. What did it cost the druggist per ounce?

15. Men's suits that had been marked to sell at a gain of 25 % were damaged and disposed of at a fire sale at half price, which was a loss of \$7.50 on each suit. What was the cost of the suits?

61. Problems involving Motion.—When an object moves, the distance that it goes in a unit of time, as a second or an hour, is called its **rate of motion, speed, or velocity**.

For example: If a train runs 60 miles in 2 hours, its velocity is *30 miles an hour*.

If the velocity does not change throughout the motion, the motion is said to be *uniform*. In case a body moves with uniform motion, the distance that it goes in any length of time is obtained by multiplying the velocity by the number of units of time; that is,

$$\text{distance} = \text{time} \times \text{velocity}.$$

EXERCISES

1. An express train runs with a velocity of 45 miles an hour. How far does it go in 4 hours?

2. The "Rocky Mountain Limited," on the Rock Island Road, runs from Omaha to Denver, a distance of 580 miles, in 14 hours and 48 minutes. What is the average speed of the train, making no allowances for stops?

3. Sound travels through the air 1080 feet a second. If a flash of lightning is a mile away, how long after the flash should one hear the thunder?

4. A train leaves a station and runs at a speed of 40 miles an hour. Two hours later a second train leaves the same station and runs over the same track at the rate of 55 miles an hour. In how many hours will the second train pass the first?

SUGGESTION.—Let t = number of hours after second train starts until it passes the first. Then, since the first train has a start of two hours, $55t = 40(t + 2)$.

5. A messenger, going 6 miles per hour, has been gone for two hours when it is found that the message is wrong. A second messenger, riding at an average of 10 miles per hour, is sent to overtake him. In how many hours will the second messenger overtake the first?

6. Two motor cyclists start at the same time from points 280 miles apart, and travel towards each other. One rides 24 miles per hour, but is delayed 3 hours on the road because of a break in his machine. The other rides 20 miles per hour uninterruptedly. In how many hours will they meet?

7. Two pedestrians started at the same time from points $44\frac{1}{2}$ miles apart, one walking at the rate of $2\frac{1}{2}$ miles an hour, and the other at the rate of $2\frac{3}{4}$ miles an hour. When and where did they meet?

8. Two automobile parties started from the same place, one going north at 20 miles an hour and the other south at 18 miles an hour. In what time will they be 120 miles apart?

9. A motor cyclist rode 75 miles in 4 hours. Part of the distance was on a country road at a speed of 20 miles an hour, and the rest within the city limits at 10 miles an hour. Find how many hours of his ride were in the country.

10. A train runs from Joliet, Ill., to Chicago, 39.6 miles, in 1.1 hours. The run from Joliet to Englewood is made at 40 miles an hour, and the run from Englewood to Chicago at 24 miles an hour. Find the time required to make the run from Englewood to Chicago. Find the distance from Englewood to Chicago.

11. A train running from Chicago to Denver at the average speed of 40 miles an hour takes three hours longer to make the run than one running at 45 miles an hour. Find the distance from Chicago to Denver.

SUGGESTION. — First find the time required at a speed of 40 miles an hour.

12. An ocean liner going 20 knots an hour leaves New York when a freighter going 6 knots an hour is already 90 knots out. How long will it take the liner to overtake the freighter?

13. A travels 8 hours at a rate of 3 miles per hour less than the rate of B. B travels an equal distance in 6 hours. What is the rate of each?

14. Two trains approach each other, leaving stations 149 miles apart at the same time. One goes 10 miles per hour faster than the other, and they meet in 2 hours. What is the rate of each?

15. The longest steel bridge in the world is said to be one of the Spokane, Portland, & Seattle Railroad, over the Columbia River. Including approaches, it is 2 miles long. It would take a freight train 1144 feet long, running 15 miles per hour (1320 feet per minute), just 3 minutes to cross the ten main spans. Find the length of the ten main spans.

16. A man shooting at a target heard the bullet strike the target $3\frac{3}{5}$ seconds after he fired. The bullet was known to travel 1375 feet a second; and sound travels approximately 1100 feet a second. How long after he fired did the bullet hit the target? How far away was the target?

17. A bullet going 1650 feet per second is heard to strike a target 2 seconds after it is fired. How long after it is fired does it hit the target? How far away is the target?

18. Find the time between 4 and 5 o'clock when the hands of a clock are together.

SUGGESTION. — Let x = the number of minute spaces which the minute hand has traveled from 4 o'clock on until it overtook the hour hand. Then $\frac{1}{12}x$ will be the number of minute spaces which the hour hand has traveled meanwhile. Why? The difference is 20. Why?

19. Find the time between 7 and 8 o'clock when the hands of a clock are together.

20. At what time between 6 and 7 o'clock is the minute hand 15 minute spaces behind the hour hand?

SUGGESTION. — Starting at 6 o'clock, how many minute spaces has the minute hand had to gain on the hour hand?

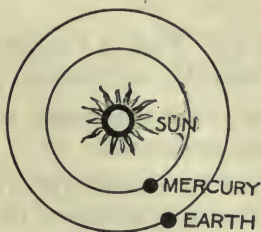
21. Find the time between 4 and 5 o'clock when the hands of a clock are directly opposite each other.



22. At what time between 8 and 9 o'clock are the hands of a clock directly opposite each other?

23. How long is it from the time that the hour and minute hands of a clock are together until they are together again?

SUGGESTION. — Let t = the number of minutes required. Since the minute hand makes one revolution in 60 minutes, it makes $\frac{1}{60}$ of a revolution in one minute, and $\frac{1}{60}t$ revolutions in t minutes. Similarly, the hour hand makes $\frac{1}{720}$ of a revolution in one minute, and $\frac{1}{720}t$ revolutions in t minutes. Hence, $\frac{1}{60}t - \frac{1}{720}t = 1$.



24. The earth makes a circuit around the sun in 12 months, and Mercury makes a circuit around the sun in 3 months. If the earth and Mercury are "in conjunction," as in the figure, how long will it be until they are in conjunction again?

25. Venus makes a circuit around the sun in 225 days, and Mars in 687 days. How many days elapse between two consecutive conjunctions of these two planets?

62. Problems on Specific Gravities of Substances. — A cubic foot of steel weighs 7.8 times as much as a cubic foot of water. This number, 7.8, is called the **specific gravity** of steel. In general,

The ratio of the weight of a given volume of any solid or liquid substance to the weight of an equal volume of water at the freezing point of temperature is the specific gravity of the substance.

What does it mean to say that the specific gravity of alcohol is .79?

A cubic centimeter of distilled water at the freezing point weighs just 1 gram. Since the specific gravity of pure gold is 19.36, 1 cubic centimeter of gold weighs 19.36 grams; 2 cubic centimeters weigh 2×19.36 grams; 3 cubic centimeters weigh 3×19.36 grams; etc. Evidently,

The weight of an object in grams equals the product of its volume in cubic centimeters and its specific gravity.

Thus, if the specific gravity of copper is 8.9, 1 cubic centimeter of copper weighs 8.9 grams. Hence, 10 cubic centimeters of copper weigh 10×8.9 grams, or 89 grams.

Certain problems in the specific gravities of mixtures of substances are solved by use of linear equations.

EXERCISES

1. The specific gravity of cast iron is 7.2. Find the weight of 10 cubic centimeters of cast iron. Of 200 cubic centimeters.

2. The specific gravity of glass is 2.89. Find the weight of 100 cubic centimeters of glass. Of 25 cubic centimeters.

3. Brass is made of copper and zinc, and its specific gravity is 8.4. How many cubic centimeters of copper, of which the specific gravity is 8.9, must be used with 200 cubic centimeters of zinc, of which the specific gravity is 6.9, to make brass?

SUGGESTION. — The volume of the brass is the sum of the volumes of the copper and zinc, and the weight of the brass is the sum of the weights of the copper and zinc. Let v = volume of copper in cubic centimeters. Then, weight of copper = $8.9v$, weight of zinc = 6.9×200 , and weight of brass = $8.4(v + 200)$. Hence, $8.9v + 6.9 \times 200 = 8.4(v + 200)$.

4. How much zinc must be combined with 250 cubic centimeters of copper to form brass?

5. How many cubic centimeters of water (specific gravity 1) must be mixed with 500 cubic centimeters of alcohol (specific gravity .79) so that the specific gravity of the mixture shall be .9?

6. A piece of ice containing 1,000,000 cubic centimeters, specific gravity .92, floats in water. How many cubic centimeters in an oak beam, specific gravity 1.17, that may be placed upon the ice without making it sink?

SUGGESTION. — The specific gravity of the combination of ice and oak must equal 1, the specific gravity of water.

7. How much steel, specific gravity 7.8, must be attached to a piece of white pine, specific gravity .42, containing 10,000 cubic centimeters, in order that the specific gravity of the combined materials may be 2?

8. In Problem 7, how much steel must be used so that the combined materials will just float in water?

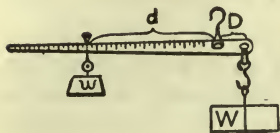
9. A piece of glass containing 850 cubic centimeters, specific gravity 2.89, is made to float by attaching cork to it. How much cork, specific gravity .24, must be used?

10. My watch case is made of 14-karat gold, specific gravity 14.88. How much pure gold, specific gravity 19.36, must be combined with 20 cubic centimeters of nickel, specific gravity 8.57, to make 14-karat gold?

11. When 100 cubic centimeters of mercury and 10 cubic centimeters of gold (specific gravity 19.36) are combined, it is found that the specific gravity of the combined materials is 14.1. Find the specific gravity of mercury.

12. It is found that when 90 cubic centimeters of copper (specific gravity 8.9) and 150 cubic centimeters of tin are combined, the specific gravity of the combination is 7.9. What is the specific gravity of tin?

63. Problems involving the Lever. — In every form of *lever*, such as the steelyard, crowbar, nutcracker, teeter board, etc., it is found that the two weights or forces exerted balance when they are placed at such distances from the point of support or fulcrum that *the products of the weights or forces by their distances from the fulcrum are equal*.



Thus, in case of the steelyard, $wd = WD$.

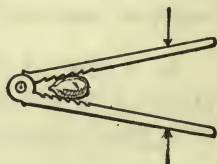
Some problems on the use of the lever may be solved by linear equations.

EXERCISES

1. On a steelyard the distance D is 2 inches and the weight w is $\frac{1}{2}$ pound. If d is 8 inches, what is the weight W ?

2. A nutcracker is held 4 inches from the hinge or fulcrum, and a nut is placed 1 inch from the hinge. A squeezing force of 3 pounds is required to crack the nut. What is its resistance?

3. Two children play at teeter. One weighs 80 pounds, and sits 5 feet from the point of support of the teeter board. If the board balances when the other sits 6 feet from the point of support, what is the weight of the second child?



4. Two boys play at teeter. One weighs 100 pounds, and sits 6 feet from the point of support. The other weighs 120 pounds. How far from the point of support must he sit in order to make the board balance?

5. A board 12 feet long is to be used as a teeter board. If the people weigh 95 pounds and 110 pounds, respectively, and sit at the ends of the board, find the point at which it must be supported in order to balance.

6. A workman lifts a stone weighing 420 pounds by means of a crowbar. If the fulcrum is placed 6 inches from the point at

which the crowbar is applied to the stone, how far from the fulcrum must the workman grasp the crowbar to lift the stone with a force of 100 pounds?

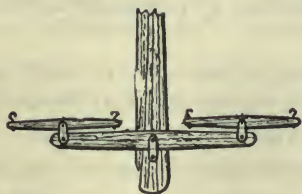
7. In Problem 6, if the man grasps the crowbar at a point 48 inches from the point where the crowbar is applied to the stone, where must the fulcrum be placed in order that he may lift the stone with a pressure of 70 pounds?



8. Weights of 240 pounds and 300 pounds, respectively, are applied at the ends of a beam 40 inches long. At what point must the beam be supported in order to balance?

9. A and B carry an object suspended from a horizontal rod 60 inches long held between them. If one is to lift only three fourths as much as the other, at what point of the rod must the object be suspended?

10. A man has a team of which one horse weighs 1200 pounds and the other 1500 pounds. If their draft power is proportional to their weight, how must he divide the 50-inch doubletree in order to justly distribute the load?



11. A farmer has a team of which one horse weighs 1400 pounds and the other 1600 pounds. If their draft power is proportional to their weight, where must he place the clevis on the 4-foot doubletree so as to justly distribute the load?

64. Problems on the Decimal Number System. — In any number such as 4258, the 8 is called the *ones' digit*, the 5 the *tens' digit*, and 2 the *hundreds' digit*, etc.

Any number of two or more digits in our decimal system of notation is in nature a polynomial. Thus, $483 = 400 + 80 + 3$, or $100 \times 4 + 10 \times 8 + 3$. Similarly, any number whose hundreds', tens', and ones' digits are respectively *h*, *t*, and *u*, may be written

in the polynomial form $100h + 10t + u$. In writing a number in the polynomial form, each digit must be multiplied by 10, 100, 1000, etc., according to the position that it occupies in the number.

EXERCISES

1. Write in the polynomial form the following numbers: 25; 347; 4196.

2. Write the number whose ones' digit is a , tens' digit b , hundreds' digit c , and thousands' digit d .

3. In a number of two digits, the sum of the digits is 9. If the digits are interchanged in position, the new number obtained exceeds the given number by 27. Find the number.

SUGGESTION. — If x = the tens' digit, $9 - x$ = the ones' digit.

Hence, $10x + 9 - x$ = the number.

Hence, $10(9 - x) + x$ = the number with the digits interchanged.

Hence, $10(9 - x) + x = 10x + 9 - x + 27$.

4. A number is composed of two digits whose sum is 12. This number exceeds by 36 the number obtained by interchanging the digits. Find the number.

5. In a certain number of two digits, the tens' digit exceeds the ones' digit by 2. The sum of the given number and that obtained by interchanging the digits is 154. What is the number?

6. A number consists of two digits of which the ones' digit is 3 more than the tens' digit, and the number is 3 more than 4 times the sum of its digits. What is the number?

7. What is the number of which the tens' digit is 2 more than the ones' digit, and which is 7 times the sum of the digits?

8. In a number of two digits the ones' digit is 3 less than the tens' digit, and the number is 20 less than twice that obtained by interchanging the digits. What is the number?

9. In a number of three digits, the tens' digit is 1 more than the ones' digit, and the hundreds' digit is 1 more than the tens' digit. The number is 21 more than 50 times the sum of its digits. Find the number.

65. Linear Equations with Two Unknown Numbers. — Problems in which the values of more than one unknown quantity are to be found are often most easily solved by use of equations containing more than one unknown number.

EXAMPLE. — A merchant sold 10 suits for \$162. He received \$15 for each of one kind and \$18 for each of the other kind. How many were there of each kind?

Let n = the number sold at \$15 each, and m = the number sold at \$18 each.

$$\text{Then} \quad \left\{ \begin{array}{l} n + m = 10, \\ 15n + 18m = 162. \end{array} \right. \quad \begin{array}{l} (1) \\ (2) \end{array}$$

$$\text{Multiplying (1) by 15,} \quad 15n + 15m = 150. \quad (3)$$

$$\text{Subtracting (3) from (2),} \quad 3m = 12. \quad (4)$$

$$\text{Hence,} \quad m = 4. \quad (5)$$

$$\text{Replacing } m \text{ by 4 in (1),} \quad n + 4 = 10. \quad (6)$$

$$\text{Hence,} \quad n = 6. \quad (7)$$

Therefore, 6 suits were sold at \$15 each and 4 suits at \$18 each.

66. Simultaneous Equations. — In the example in § 65, we have seen two equations concerning two unknown numbers whose values satisfy both equations.

Thus, in the example, when $n = 6$ and $m = 4$, equation (1) becomes $6 + 4 = 10$, and equation (2) becomes $90 + 72 = 162$.

Two or more equations containing two or more unknown numbers which satisfy all of the equations are called **simultaneous**.

67. Systems of Simultaneous Equations. — Two or more simultaneous equations are said to constitute a **system**. To solve a **system** of equations is to find the *sets of values* of the unknown numbers which satisfy all of the equations.

Thus, in the system of equations in § 65, the set of values $n = 6$ and $m = 4$ constitute a *solution*.

68. Elimination. — The solution of a system of two linear equations is most easily found by so combining the two equations as to obtain a new equation in which one of the unknown numbers does not appear. This process is called **elimination**. A sim-

ple method of elimination is shown in the example in § 65. Compare the steps in the process of solving the following system with those in the example in § 65.

EXAMPLE. — Solve the system $\begin{cases} A + 4 B = 12, \\ 2 A - B = 6. \end{cases}$ (1)

Let us first eliminate B .

Multiplying (2) by 4, $8 A - 4 B = 24.$ (3)

Adding (3) to (1), $9 A = 36.$ (4)

Hence, $A = 4.$ (5)

The value of B may now be found in like manner by eliminating A between equations (1) and (2), but is more easily found by replacing A by its value 4 in equation (1) or equation (2).

From (1), when $A = 4$, $4 + 4 B = 12.$ (6)

Whence, $B = 2.$ (7)

See if the solution $A = 4$, $B = 2$ satisfies both equations of the system.

A study of this example and that in § 65 will reveal the following rule:

To eliminate one of the unknown numbers, multiply the members of each equation, if necessary, by such a number as will make the numerical coefficients of that unknown number the same in both of the resulting equations. Add or subtract (according to signs) the corresponding members of the resulting equations.

After finding the value of one unknown number, substitute its value in either of the given equations and solve the resulting equation for the value of the other unknown number.

NOTE. — Other methods of elimination will be discussed in Chapter XII.

EXERCISES

Solve:

1. $\begin{cases} a + b = 7, \\ a - b = 3. \end{cases}$

4. $\begin{cases} 4x - 3y = 1, \\ 3x - 4y = 6. \end{cases}$

2. $\begin{cases} 5M + 4N = 22, \\ 3M + N = 9. \end{cases}$

5. $\begin{cases} 2n + 5k = 15, \\ 3n - 4k = 11. \end{cases}$

3. $\begin{cases} 6E - d = 10, \\ 7E - 2d = 15. \end{cases}$

6. $\begin{cases} P + 4R = 12, \\ 2P - R = 6. \end{cases}$

$$7. \quad \begin{cases} 7z + c = 42, \\ 3z - c = 8. \end{cases}$$

$$10. \quad \begin{cases} 2m - n - 5 = 0, \\ 5m - 2n = 14. \end{cases}$$

$$8. \quad \begin{cases} 6T + 7H = -33, \\ 5T - 4H = 2. \end{cases}$$

$$11. \quad \begin{cases} 5t - 2a = 1, \\ 8t = 5a - 11. \end{cases}$$

$$9. \quad \begin{cases} 5W - 2V - 35 = 0, \\ 4W - V - 25 = 0. \end{cases}$$

$$12. \quad \begin{cases} 10G - 11v + 27 = 0, \\ 10v - 11G = 36. \end{cases}$$

69. Problems solved by Use of Systems of Equations.—In a problem in which the values of *two* unknown quantities are sought it always will be found that the problem expresses or implies *two* facts or suppositions. Represent each unknown number by a letter. Then each of the facts or suppositions expressed in the problem gives an equation containing the unknown numbers. These two equations constitute a system, which may be solved by the method of § 68.

Many of the problems in the exercises in the earlier part of this chapter might have been solved by use of two equations containing two unknown numbers.

EXAMPLE.—I have \$1.50 in dimes and nickels. There are 25 coins in all. How many dimes and how many nickels are there?

Let d = number of dimes, and n = number of nickels.

$$\text{Then,} \quad \begin{cases} d + n = 25, & (1) \\ 10d + 5n = 150. & (2) \end{cases}$$

Note that the fact that there are 25 coins in all gives equation (1), and the fact that their total value is \$1.50 gives equation (2).

Solving this system, $d = 5$, and $n = 20$.

Hence, there are 5 dimes and 20 nickels.

EXERCISES

Solve by using two equations with two unknown numbers:

1. A man invested \$5600, part in bonds that paid 4%, and the remainder in a business enterprise that yielded 8% on the investment. The total yearly earnings from the two investments amounted to \$328. Find the amount of each investment.

SUGGESTION. — Let a = amount of investment in bonds, and b = amount of investment in business enterprise.

$$\text{Then,} \quad \begin{cases} a + b = 5600, \\ .04a + .08b = 328. \end{cases}$$

2. A man had \$2150 to invest. Part of this sum he loaned at 6% interest, and the remainder he invested in stock of a building and loan association that paid 7% dividends. From the two investments his annual income was \$142.50. What was the amount of each investment?

3. A money drawer contains \$3.50 in dimes and nickels. There are 50 coins in all. How many coins of each kind are there?

4. In playing teeter, two boys use a board 12 feet long. One boy weighs 80 pounds and the other 110 pounds. At what point must the board be supported to balance?

5. A teamster has a team of which one horse weighs 1250 pounds and the other 1500 pounds. Assuming that draft power is proportional to weight, how should he divide the 50-inch double-tree in order to properly distribute the load?

6. A merchant mixes 28-cent coffee and 36-cent coffee to sell at 30 cents a pound. What quantities of the two grades of coffee should he take to make 40 pounds of the mixture?

7. The specific gravity of copper is 8.9, that of zinc 6.9, and that of brass 8.4. Find the number of cubic centimeters of copper and the number of cubic centimeters of zinc that must be combined to produce 600 cubic centimeters of brass.

SUGGESTION. — Let c cubic centimeters copper and z cubic centimeters zinc be the amounts required.

$$\text{Then,} \quad \begin{cases} c + z = 600, \\ 8.9c + 6.9z = 8.4(c + z). \end{cases}$$

8. How much gold, specific gravity 19.36, and how much nickel, specific gravity 8.54, must be combined to make 10 cubic centimeters of an alloy of which the specific gravity is 12?

9. In a number of two digits, the sum of the digits is 10, and when the digits are interchanged, the number is increased by 36. Find the number.

10. A belt runs over two pulleys. The circumference of one pulley is 2 feet more than the circumference of the other. One pulley makes 3 revolutions while the other makes 2. Find the circumference of each pulley.

11. When the Panama Canal was finished, the distance from New York to San Francisco by boat was reduced by 7796 miles. The distance by the old route exceeded twice the distance by the Panama route by 2502 miles. Find the distance by the old route and the distance by the Panama route.

12. The distance from New York to Yokohama, Japan, was reduced by 2952 miles when the Panama Canal was completed. If the distance by the old route were twice as great, and the distance by the Panama Canal three times as great, the latter would exceed the former by 4184 miles. Find the distance by each route.

13. The ship *Mauretania*, built in 1907, is 583 feet longer than the ship *Britannica*, built in 1840. The *Mauretania* is 169 feet more than three times as long as the *Britannica*. Find the length of each.

14. In 1847 the *Deutschland*, the first ship of the Hamburg-American Line, arrived in New York. The present *Deutschland* carries 15,283 tons more than the first *Deutschland*. The tonnage of the present *Deutschland* exceeds by 226 tons 22 times the tonnage of the old *Deutschland*. Find the tonnage of each.

15. It is stated that the daily ration for a laboring man should contain 4 ounces each of fat and protein. In pork the per cent of fat is 26 and the per cent of protein is 13; and in beans the per cent of fat is 2 and the per cent of protein is 22. How many ounces each of pork and beans are required to make a ration for one day?

SUPPLEMENTARY EXERCISES

NOTE. — As we have seen, many practical formulæ contain two or more unknown numbers. It often is found necessary to solve such formulæ for one of the unknown numbers *in terms of the others*. Thus, the circumference of a circle is computed by the formula $C = 2\pi R$. If this is solved for R in terms of C , we get $R = \frac{C}{2\pi}$. Such formulæ are solved by the same processes as equations with one unknown number.

1. In the simple interest formula, $I = prt$, solve for t in terms of I , p , and r .

2. The distance an object moves in t seconds at v feet per second is $s = tv$. Solve for t .

3. The area of a triangle is expressed by $A = \frac{1}{2}BH$. Solve for H .

4. The area of a trapezoid is expressed by $A = \frac{1}{2}H(B + B')$. Solve for B' .

5. The relation between the readings on the Fahrenheit and Centigrade thermometers for any temperature is expressed by $C = \frac{5}{9}(F - 32)$. Solve for F .

6. The formula for finding the horse power of a steam engine is $H. P. = \frac{plan}{33000}$, where p = the mean effective pressure in pounds per square inch, l = the length of stroke in feet, a = area of piston in square inches, n = twice the number of revolutions. Solve for l , the length of stroke.

7. In a polygon with n sides the sum of all the angles, in degrees, is given by $s = 180(n - 2)$. Solve for n .

8. Solve $2ax - b = cy$ for x .

9. Solve $A(n - 1) + A = n$ for A .

10. Solve $(V - S)(V + 2S) = V^2 + S^2$ for V .

11. Solve $2(2t - 1) + 3 = a(t + 2)$ for t .

12. Solve $R(R + k) - k(1 - R) = R^2 + 1$ for R .

In each of the following systems solve for x and y :

$$13. \quad \begin{cases} 8x + y = 60a, \\ 7x - 10y = 9a. \end{cases}$$

$$15. \quad \begin{cases} 3x + 7by = 16b, \\ 2x + 5by = 13b. \end{cases}$$

$$14. \quad \begin{cases} x + y = 30n, \\ 3x - 2y = 25n. \end{cases}$$

$$16. \quad \begin{cases} 5mx - 2ny = 63mn, \\ 2mx + ny = 18mn. \end{cases}$$

17. A square has the same area as a rectangle whose length is 3 inches greater and width 2 inches less than the side of the square. Find the dimensions of the square and of the rectangle.

18. In a certain family each daughter has as many sisters as brothers; but each son has twice as many sisters as brothers. How many children are in the family?

19. The leader in a "guessing game" tells each of the others to add 6 years to his age, multiply the result by 4, subtract 24 from the product, and to the remainder add his age. When the results are announced, the leader tells at once the age of each individual. If an individual gives 80 as the result, what is his age?

20. Find the longitude of a place in the Central Time belt at which it is observed that the local or sun time is 15 minutes faster than standard time.

21. Find the longitude of a place in the New York Time belt whose local time is 10 minutes slower than standard time.

22. A crop of 60 bushels per acre of corn takes from one acre of soil 58 pounds less of phosphoric acid than of nitrogen, and 23 pounds less of potash than of nitrogen. Five times the amount of phosphoric acid exceeds twice the amount of potash by 8 pounds. How many pounds of each of these fertilizers are taken from an acre of soil?

23. Fifteen pounds of tin weigh 13 pounds in water, and 15 pounds of zinc weigh 13.5 pounds in water. How much tin and how much zinc in an alloy which weighs 56 pounds in air and 49 pounds in water?

NOTE. — The method given in this problem of finding the proportional parts of metals in an alloy by first weighing the pure metals and the alloy

each in air, then in water, is said to have been first discovered and used by Archimedes, about 220 B.C., in determining for King Hieron of Syracuse whether a crown, claimed by the maker to be pure gold, was not alloyed with silver.

24. Two seconds after a marksman fires he hears the bullet hit the target, which is 440 yards distant. If sound travels through the air at a velocity of 1100 feet a second, find the average velocity of the bullet.

25. Two weights balance when one is 12 inches and the other 15 inches from the point of support. If the first is replaced by a weight 6 pounds greater, the second must be moved 3 inches farther from the point of support to balance it. Find the weights.

~~26.~~ 26. A crew that can row 6 miles an hour down a stream can row 2 miles an hour up the stream. Find the speed of the current and the speed at which the crew can row in still water.

27. A local train 820 feet long and an express train 500 feet long run on parallel tracks. When running in the same direction it requires 88 seconds for the express to pass the local, but when running in opposite directions it requires only 17.6 seconds for them to pass. Find the rates of the two trains.

CHAPTER VII

SPECIAL PRODUCTS AND QUOTIENTS

70. Special Rules.—The products or quotients of expressions of many forms, called **type forms**, may be written down at sight by special rules, without performing in detail the complete processes of multiplication or division. By discovering and learning these rules, much time and labor in multiplications and divisions may be saved. They reveal also many “short cuts” in the multiplication of arithmetical numbers that are valuable to know, some of the most important of which are given in this chapter. The need for use of these special rules presents itself so often in algebra that they should be thoroughly mastered before proceeding farther.

71. Squares and Cubes of Monomials.—What is the product of $a^3 \times a^3$? Of $5m^4n^5 \times 5m^4n^5$? Of $(-3xy^2z^6)(-3xy^2z^6)$?

Since $(7M^4)^2$ means $7M^4 \times 7M^4$, what is its value? Find the value of $(9P^2Q^5)^2$. Of $(-4h^3k^8)^2$.

From these examples, the following rule is evident:

To square a monomial, square the numerical coefficient, and multiply each exponent by 2.

Since $(2w^2)^3$ means $2w^2 \times 2w^2 \times 2w^2$, what is its value? Find the value of $(3H^3t^4)^3$. Of $(-4A^2B^4C^6)^3$.

These examples reveal the following rule:

To cube a monomial, cube the numerical coefficient, and multiply each exponent by 3.

Similar rules may be discovered for raising monomials to the fourth power, the fifth power, etc.

EXERCISES

Give at sight the values of the indicated powers :

- | | | | |
|------------------------|-------------------------|-----------------------------|-------------------------|
| 1. $(a^4)^2$. | 11. $(L^4)^3$. | 21. $(x^4y^2)^2$. | 31. $(a^4s^7v^3)^3$. |
| 2. $(R^6)^2$. | 12. $(t^5)^3$. | 22. $(ab^5)^2$. | 32. $(3x^2y^5)^2$. |
| 3. $(w^{10})^2$. | 13. $(S^8)^3$. | 23. $(y^2z^6)^2$. | 33. $(5MN^4)^2$. |
| 4. $(k^5)^2$. | 14. $(x^6)^3$. | 24. $(r_1^2r_2^2r_3^2)^2$. | 34. $(2a^2b^4c^6)^2$. |
| 5. $(p^7)^2$. | 15. $(r^7)^3$. | 25. $(W^6g^2)^2$. | 35. $(8w^{10}v^2)^2$. |
| 6. $(n^{12})^2$. | 16. $(Q^{12})^3$. | 26. $(H^4p^7)^2$. | 36. $(11AN^8)^2$. |
| 7. $(B^8)^2$. | 17. $(v^{10})^3$. | 27. $(a^2t^3)^3$. | 37. $(15xy^4z^7)^2$. |
| 8. $(V^3)^2$. | 18. $(N^{15})^3$. | 28. $(V^4y^4)^3$. | 38. $(6v^3t^4)^2$. |
| 9. $(x^9)^2$. | 19. $(p^{20})^3$. | 29. $(A^3n^5)^3$. | 39. $(9S^5H^2)^2$. |
| 10. $(A^{15})^2$. | 20. $(b^{16})^3$. | 30. $(w^6n^4)^3$. | 40. $(16m^2n^8p^3)^2$. |
| 41. $(3D^3W^6)^3$. | 48. $(-8E^5H^6)^2$. | 55. $(-3B^6T^2)^3$. | |
| 42. $(5x^4T)^3$. | 49. $(-2a^9b^9c^9)^2$. | 56. $(-4H^2g^7v^9)^3$. | |
| 43. $(2a^3n^2b^4)^3$. | 50. $(-10RS^4V^4)^2$. | 57. $(-2P^5V^2R)^3$. | |
| 44. $(4p^2t^4y)^3$. | 51. $(-M^6W^6)^2$. | 58. $(-3L^2T^4)^3$. | |
| 45. $(6V^3n^2g^2)^3$. | 52. $(-3a^5b^4c^3)^2$. | 59. $(-2w^2v^2)^3$. | |
| 46. $(-2A^4)^2$. | 53. $(-x^2y^2)^3$. | 60. $(-3a^3)^2(2a^2)^3$. | |
| 47. $(-3m^3n^7)^2$. | 54. $(-2a^2t^4)^3$. | 61. $(2n^4)^3(-3n^5)^2$. | |

72. Square Roots and Cube Roots of Monomials.—What is the number which squared gives 9? 25? a^2 ?

The number whose square is a given number is called the **square root** of the given number.

Thus, since $7^2 = 49$, the square root of 49 is 7.

Similarly, the number whose cube is a given number is called the **cube root** of the given number.

Thus, since $(2v^2)^3 = 8v^6$, the cube root of $8v^6$ is $2v^2$.

The numbers whose fourth powers, fifth powers, etc., are a given number are called the **fourth root**, **fifth root**, etc., of the given number.

A root of a number is indicated by placing before it the sign $\sqrt{}$, called the **radical sign**. Usually a vinculum is attached to the radical sign, to show how far its effect is to extend. To indicate what root it is, except in the case of a square root, a little number called the **index** is written above the radical sign.

For example, the square root of n is written $\sqrt{n}$; the cube root of n is written $\sqrt[3]{n}$; the fourth root of n is written $\sqrt[4]{n}$; etc. In the case of square root the index 2 is *understood* and left off.

In most simple practical work, one needs only a knowledge of square roots and cube roots.

Since $(+3)^2=9$ and $(-3)^2=9$, $\sqrt{9}$ is either $+3$ or -3 . Similarly, since $(+6)^2=36$ and $(-6)^2=36$, $\sqrt{36}$ is either $+6$ or -6 . In general,

Every positive number has two square roots, which have the same absolute value, one positive and the other negative.

The two square roots of a positive number are often written together, with a **double sign** $\pm$.

Thus, $\sqrt{81} = \pm 9$; $\sqrt{144} = \pm 12$.

Since no positive or negative numbers squared can give a negative number,

A negative number has no square root that can be expressed as a positive or negative number.

NOTE. — What to do in a problem where the square root of a negative number is required to be found will be discussed later.

Since $(+2)^3=8$ and $(-2)^3=-8$, $\sqrt[3]{8}=2$ and $\sqrt[3]{-8}=-2$. In general,

Any number has only one cube root that can be expressed as a positive or negative number. This cube root of a positive number is positive, and of a negative number it is negative.

NOTE. — It will be shown later that every number has, in addition to the one positive or negative cube root, two other cube roots that cannot be expressed as simple positive or negative numbers. In this chapter only the positive or negative cube root of a number will be considered.

Since the square root of a monomial, when squared, must give that monomial, it follows from § 71 that:

To find the square roots of a monomial, take the square roots of the numerical coefficient and divide the exponent of each letter by 2.

Similarly, it follows, from § 71, that:

To find the cube root of a monomial, take the cube root of the numerical coefficient, and divide the exponent of each letter by 3.

Thus, $\sqrt{64 a^6 b^4} = \pm 8 a^3 b^2$; $\sqrt{100 x^3 y^2} = \pm 10 x^{\frac{3}{2}} y$. And $\sqrt[3]{8 w^6 t^3} = 2 w^2 t$; $\sqrt[3]{-27 m^{12} n^6} = -3 m^4 n^2$.

EXERCISES

Give at sight the square roots of:

1. 1, 4, 16, 81, 121, 49, 144, 36.
2. a^4 , W^8 , x^6 , P^{20} , n^{16} , A^{12} , y^{10} .
3. M^{14} , D^{22} , S^{18} , k^{32} , $9 x^6$, $16 a^4$.
4. $49 B^{10}$, $25 v^8$, $100 R^{12}$, $36 a^4 b^6$, $81 x^8 y^2$.
5. $16 W^6 V^{12}$, $9 m^8 n^2$, $25 d^6 t^4$, $121 k^4 p^4$, $225 r_1^4 r_2^8$.
6. $144 a^6 t^{10}$, $49 w^{14} d^8$, $169 x^4 y^6 z^2$, $625 R^4 m^{16}$.
7. $196 a^{10} b^2 c^4$, $256 H^8 T^8 a^2$, $441 A^{14} x^{20} y^6$, $400 m^{40} n^{30}$.

Give at sight the cube roots of:

8. 1, 8, 27, 64, 125, 216, 512, 1000.
9. -1, -8, -27, -64, -125, -1000.
10. A^6 , n^{12} , t^3 , W^{15} , x^{21} , P^{36} , r^{18} .
11. v^{21} , y^9 , M^{24} , a^{33} , $-D^9$, $-z^{12}$, $-L^6$.
12. $-w^{18}$, $-T^{36}$, $-8 x^6$, $-27 b^{18}$, $-125 P^9$.
13. $64 R^6$, $27 v^6 t^9$, $-8 m^{12} n^3$, $-125 r_1^6 r_2^6$, $-A^{12} t^9 d^3$.
14. $-p^3 q^6 r^9$, $216 M^9 a^{15}$, $-27 x^{12} y^9 z^6$, $343 V^9 V^6$, $-8 F^3 D^{12}$.

15. The area of a square is $49d^2$. What is the length of one side?

16. The base and altitude of a right triangle are $8x$ and $6x$, respectively. Find the length of the hypotenuse.



SUGGESTION. — The square of the hypotenuse equals the sum of the squares of the base and altitude.

17. The volume of a cube is $64E^3$. Find the length of one edge.

18. The diameter D , in inches, necessary of a steel shaft, upon which are fastened pulleys that drive machines in a factory, in order to impart H horse power to the machinery when making N revolutions a minute, is computed by the formula $D = \sqrt[3]{\frac{80 \times H}{N}}$.

Find the diameter of the shaft necessary to impart 80 horse power when making 100 revolutions a minute.

73. Equations solved by finding Square Roots. — Many equations may be solved by extracting roots. The following illustrate the solutions of equations of the second degree by finding square roots.

EXAMPLE 1. — Solve $P^2 = 9$.

$$\begin{aligned} \text{Since} \quad P^2 &= 9, P = \sqrt{9} \\ &= \pm 3. \end{aligned}$$

EXAMPLE 2. — Solve $y^2 - 49 = 0$.

$$\begin{aligned} \text{Transposing,} \quad y^2 &= 49. \\ \text{Hence,} \quad y &= \sqrt{49} \\ &= \pm 7. \end{aligned}$$

EXERCISES

Solve:

- | | | |
|---------------------|----------------------|-----------------------|
| 1. $x^2 = 25$. | 6. $R^2 - 64 = 0$. | 11. $3p^2 - 48 = 0$. |
| 2. $t^2 = 81$. | 7. $B^2 - 121 = 0$. | 12. $7D^2 - 28 = 0$. |
| 3. $A^2 = 36$. | 8. $F^2 - 225 = 0$. | 13. $49 = v^2$. |
| 4. $n^2 = 100$. | 9. $2a^2 = 8$. | 14. $144 = S^2$. |
| 5. $y^2 - 16 = 0$. | 10. $5M^2 = 45$. | 15. $75 = 3W^2$. |

16. The time required for an object starting from rest to fall a given distance is found from the equation $16 t^2 = s$, where t is the time in seconds and s is the distance in feet.

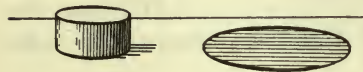
Find the time required for an object to fall 64 feet.

SUGGESTION. — $16 t^2 = 64$. Solve for t . Has the negative answer any meaning?

17. In the following table are given the distances that an object falls. Complete the table by finding, by the formula in Problem 16, the time required in each case.

DISTANCE	16 Ft.	144 Ft.	576 Ft.	1600 Ft.
Time				

18. In the manufacture of lids for metal boxes and cans, a circular piece of metal, called the "blank," is cut from a flat sheet of it, and then stamped into the required shape by means of a machine called a "die." In computing the size



of the blank to be cut for a lid of given size, men assume that the area of the blank is equal to the total area of the lid.

Find the radius of the blank necessary to make a lid whose area is 12.5664 sq. in.

SUGGESTION. — If R is the radius, $3.1416 R^2 = 12.5664$.

19. The area of the surface of the lid of a lard bucket is 28.2744 square inches. Find the radius of the blank from which it is made.

74. **Product of the Sum and Difference of Two Terms.** — Find, as in Chapter V, the products: $(a - 4)(a + 4)$; $(P + 5)(P - 5)$; $(x^2 - 7)(x^2 + 7)$; $(2 R^3 + 3)(2 R^3 - 3)$.

How many terms in each product? Why not more? Why is the last term negative in each product?

In general, where a and b are any two terms, it is found that:

$$(a + b)(a - b) = a^2 - b^2.$$

Hence,

To find the product of the sum and the difference of the same two terms, take the difference between their squares.

$$\begin{aligned}\text{EXAMPLE. — } (4w^3 + 3)(4w^3 - 3) &= (4w^3)^2 - 3^2 \\ &= 16w^6 - 9.\end{aligned}$$

EXERCISES

Give at sight the products of :

- | | |
|-------------------------------|--|
| 1. $n + 3$ and $n - 3$. | 18. $(12H - 5a)(12H + 5a)$. |
| 2. $A + 6$ and $A - 6$. | 19. $(a^2 - 9)(a^2 + 9)$. |
| 3. $v + 2$ and $v - 2$. | 20. $(x^3 - 4)(x^3 + 4)$. |
| 4. $x + 5$ and $x - 5$. | 21. $(2A^2 - 1)(2A^2 + 1)$. |
| 5. $k - 4$ and $k + 4$. | 22. $(7s^2 - 4t^2)(7s^2 + 4t^2)$. |
| 6. $n - 7$ and $n + 7$. | 23. $(1 - 6x^4)(1 + 6x^4)$. |
| 7. $w + v$ and $w - v$. | 24. $(a^3 - b^3)(a^3 + b^3)$. |
| 8. $2a + 3$ and $2a - 3$. | 25. $(x^4 - y^4)(x^4 + y^4)$. |
| 9. $5P + 1$ and $5P - 1$. | 26. $(P^6 + Q^6)(P^6 - Q^6)$. |
| 10. $4S - 7$ and $4S + 7$. | 27. $(4 - A)(A + 4)$. |
| 11. $10 - 3R$ and $10 + 3R$. | 28. $(1 - 6x)(6x + 1)$. |
| 12. $1 - x^2$ and $1 + x^2$. | 29. $(2M + 7N)(-7N + 2M)$. |
| 13. $6 + 2v$ and $6 - 2v$. | 30. $(12 + 5t^2)(-12 + 5t^2)$. |
| 14. $2x + 5z$ and $2x - 5z$. | 31. $(-10D^3 + 9)(10D^3 + 9)$. |
| 15. $9M + 3T$ and $9M - 3T$. | 32. $(a + 1)(a - 1)(a^2 + 1)$. |
| 16. $5a - 4b$ and $5a + 4b$. | 33. $(2 - W)(2 + W)(4 + W^2)$. |
| 17. $(6W + 7V)(6W - 7V)$. | 34. $(1 - x)(1 + x)(1 + x^2)(1 + x^4)$. |

Give at sight the quotients of the following :

- | | |
|----------------------------------|---------------------------------|
| 35. $(a^2 - b^2) \div (a + b)$. | 37. $(D^2 - 1) \div (D - 1)$. |
| 36. $(x^2 - y^2) \div (x - y)$. | 38. $(16 - R^2) \div (4 - R)$. |

39. $(9h^2 - 25t^2) \div (3h + 5t)$. 42. $(144n^6 - 1) \div (12n^3 - 1)$.
 40. $(81A^4 - 1) \div (9A^2 - 1)$. 43. $(64S^4 - 25) \div (5 + 8S^2)$.
 41. $(y^4 - 36) \div (y^2 + 6)$. 44. $(36x^2 - 49y^4) \div (6x - 7y^2)$.

Solve without the aid of pencil:

45. $n^2 + n - 6 = (n - 2)(n + 2)$.
 46. $x^2 + 2x + 3 = (x + 3)(x - 3)$.
 47. $4A^2 - A + 1 = (2A + 7)(2A - 7)$.
 48. $9R(R + 2) = (3R + 6)(3R - 6)$.

75. Products of Arithmetical Numbers by the Rule in § 74.—The product of two arithmetical numbers, of which one is less than a multiple of 10 and the other exceeds this multiple by the same amount, may be given at sight by the rule in § 74.

Thus, $58 \times 62 = (60 - 2)(60 + 2) = 60^2 - 2^2 = 3600 - 4 = 3596$.

EXERCISES

Find mentally the products of the following:

- | | | |
|---------------------|----------------------|------------------------|
| 1. 19×21 . | 7. 48×52 . | 13. 85×75 . |
| 2. 28×32 . | 8. 53×47 . | 14. 88×92 . |
| 3. 22×18 . | 9. 69×71 . | 15. 99×101 . |
| 4. 39×41 . | 10. 72×68 . | 16. 97×103 . |
| 5. 43×37 . | 11. 63×57 . | 17. 109×111 . |
| 6. 34×26 . | 12. 77×83 . | 18. 118×122 . |
19. Find the cost of 22 boxes of berries at 18 cents a box.
 20. Find the cost of 17 dozen eggs at 23 cents a dozen.
 21. Find the cost of 28 pounds of butter at 32 cents a pound.
 22. Buckwheat weighs 52 pounds a bushel. Find the weight of 48 bushels.
 23. Onions weigh 57 pounds a bushel. Find the weight of 63 bushels.
 24. A field is 77 rods wide and 83 rods long. Find its area.

76. Product of Two Binomials with One Common Term.

$x+7$	$x-7$	$x+7$	$x-7$
$x+5$	$x-5$	$x-5$	$x+5$
$\hline x^2+7x$	$\hline x^2-7x$	$\hline x^2+7x$	$\hline x^2-7x$
$+5x+35$	$-5x+35$	$-5x-35$	$+5x-35$
$\hline x^2+12x+35$	$\hline x^2-12x+35$	$\hline x^2+2x-35$	$\hline x^2-2x-35$

Tell how the first term of each of the above products is obtained. How is the coefficient in the second term of each obtained? How is the third term of each obtained?

In general,

$$(x+a)(x+b) = x^2 + (a+b)x + ab.$$

That is,

To find the product of two binomials having a common term, take the square of the common term, plus the algebraic sum of the unlike terms times the common term, plus the algebraic product of the unlike terms.

EXAMPLE 1. — $(n+2)(n+3) = n^2 + (2+3)n + 2 \cdot 3$
 $= n^2 + 5n + 6.$

EXAMPLE 2. — $(t-7)(t+4) = t^2 + (-7+4)t + (-7 \cdot 4)$
 $= t^2 - 3t - 28.$

EXERCISES

Give at sight the products of:

- | | | |
|------------------|--------------------|---------------------|
| 1. $(a+4)(a+5).$ | 10. $(m+7)(m+8).$ | 19. $(d-15)(d-2).$ |
| 2. $(x+1)(x+3).$ | 11. $(K+1)(K+10).$ | 20. $(S-6)(S-12).$ |
| 3. $(M+2)(M+6).$ | 12. $(x+8)(x+6).$ | 21. $(v-4)(v-10).$ |
| 4. $(y+7)(y+4).$ | 13. $(a-5)(a-3).$ | 22. $(d-16)(d-10).$ |
| 5. $(n+6)(n+1).$ | 14. $(W-8)(W-2).$ | 23. $(a+5)(a-3).$ |
| 6. $(v+3)(v+8).$ | 15. $(R-6)(R-1).$ | 24. $(x+12)(x-8).$ |
| 7. $(t+4)(t-9).$ | 16. $(t-3)(t-12).$ | 25. $(n+8)(n-4).$ |
| 8. $(A+7)(A+3).$ | 17. $(B-16)(B-3).$ | 26. $(V-3)(V+9)$ |
| 9. $(D+5)(D+5).$ | 18. $(N-9)(N-7).$ | 27. $(t-5)(t+8).$ |

28. $(b-12)(b+7)$. 34. $(m+16)(m-12)$. 40. $(y^4+7)(y^4-11)$.
 29. $(x+3)(x-15)$. 35. $(u-11)(u+9)$. 41. $(c^2-10)(c^2+4)$.
 30. $(y-1)(y+6)$. 36. $(C+18)(C-7)$. 42. $(2a+5)(2a+3)$.
 31. $(H+4)(H-9)$. 37. $(x^2+2)(x^2+5)$. 43. $(3x+7)(3x+2)$.
 32. $(R-15)(R+10)$. 38. $(a^3-1)(a^3+9)$. 44. $(6t-1)(6t-4)$.
 33. $(A+14)(A-4)$. 39. $(P^2-6)(P^2-8)$. 45. $(8x+3)(8x-2)$.
 46. $(4W-2)(4W+5)$. 55. $(6k+3t)(6k-5t)$.
 47. $(5y+7)(5y-2)$. 56. $(4m+9n)(4m-3n)$.
 48. $(2c-3)(2c-12)$. 57. $(9D-3v)(9D+8v)$.
 49. $(8v+16)(8v-9)$. 58. $(7v+2w)(7v-9w)$.
 50. $(2x^2+5)(2x^2+3)$. 59. $(3a^2+4b^2)(3a^2+12b^2)$.
 51. $(7A^3-2)(7A^3+6)$. 60. $(5t^2-6s^2)(5t^2+4s^2)$.
 52. $(12W^2-20)(12W^2+16)$. 61. $(8B^2-4AC)(8B^2+AC)$.
 53. $(2a+3b)(2a+4b)$. 62. $(4c^2-22d^2)(4c^2+5d^2)$.
 54. $(5x-2y)(5x-6y)$. 63. $(1-5a)(1+3a)$.

Find mentally the indicated quotients:

64. $(x^2+5x+6) \div (x+2)$. 69. $(s^2+4s-60) \div (s+10)$.
 65. $(n^2-4n+3) \div (n-1)$. 70. $(W^2+W-72) \div (W-8)$.
 66. $(A^2-2A-15) \div (A+3)$. 71. $(y^2+9y+20) \div (y+4)$.
 67. $(v^2-7v+12) \div (v-4)$. 72. $(r^2-16r+63) \div (r-9)$.
 68. $(T^2-T-12) \div (T-4)$. 73. $(m^2-3m-54) \div (m+6)$.

Solve mentally:

74. $(x+3)(x+4) = x^2 + 33$.
 75. $(v+5)(v-3) = (v+2)(v-1)$.
 76. $(2n+7)(2n-6) = 4(n^2+2)$.

77. Products of Arithmetical Numbers by the Rule of § 76. — The products of many arithmetical numbers may be obtained, without the aid of pencil, by the rule of § 76.

$$\begin{aligned}\text{Thus, } 87 \times 92 &= (90 - 3)(90 + 2) = 90^2 - 1 \times 90 - 6 \\ &= 8100 - 90 - 6 \\ &= 8004.\end{aligned}$$

EXERCISES

Find mentally the following indicated products:

- | | | |
|---------------------|----------------------|------------------------|
| 1. 32×33 . | 6. 43×38 . | 11. 89×96 . |
| 2. 51×52 . | 7. 48×53 . | 12. 87×91 . |
| 3. 42×43 . | 8. 76×74 . | 13. 112×113 . |
| 4. 64×66 . | 9. 87×83 . | 14. 198×199 . |
| 5. 39×42 . | 10. 98×92 . | 15. 251×253 . |

16. Find the cost of 34 dozen eggs at 32 cents a dozen.

17. If a train goes 42 miles an hour, how far will it go in 48 hours?

18. Find the cost of 34 yards of cloth at 36 cents a yard.

19. How many square feet in a lot 92 feet wide and 98 feet long?

20. Oats weigh 32 pounds to the bushel. Find the weight of 35 bushels.

78. Square of a Binomial. — What operation is indicated by $(n + 6)^2$? Find, by multiplication, the values of:

$$(x + 3)^2; \quad (A - 5)^2; \quad (m + n)^2; \quad (m - n)^2.$$

What two terms in the square are positive in each of the above cases? When is the second term of the square negative?

It is seen that for any values of a and b ,

$$(a + b)^2 = a^2 + 2ab + b^2 \text{ and } (a - b)^2 = a^2 - 2ab + b^2.$$

These identities give the following rule :

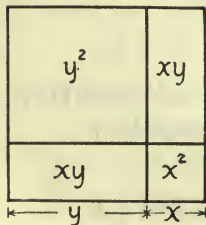
To square a binomial take the square of the first term, plus (or minus) two times the product of the terms, plus the square of the second term.

$$\begin{aligned}\text{EXAMPLE. — } (2a^2 - 3b)^2 &= (2a^2)^2 - 2(2a^2)(3b) + (3b)^2 \\ &= 4a^4 - 12a^2b + 9b^2.\end{aligned}$$

EXERCISES

Find the squares of the following :

- | | | |
|-----------------|------------------|---------------------|
| 1. $n + 3$. | 12. $7w - 3v$. | 26. $x^2 - y^2$. |
| 2. $x + 5$. | 13. $5B + 9T$. | 27. $A^2 + 2B^2$. |
| 3. $P + 2$. | 14. $10b - c$. | 28. $3t^2 - 4s^2$. |
| 4. $a - 4$. | 15. $9F + 8C$. | 29. $5w^3 + 2v^3$. |
| 5. $W - 6$. | 16. $4K - 7L$. | 30. $y^4 + 1$. |
| 6. $t - 3$. | 17. $12x + y$. | 31. $4k^2 - t$. |
| 7. $m + n$. | 18. $A - 15B$. | 32. $z^6 + 5$. |
| 8. $x - y$. | 19. $11W + 3a$. | 33. $12 - 7a^2$. |
| 9. $2a + 3b$. | 20. $4R - 9n$. | 34. $9 + 15R^3$. |
| 10. $5r - 4s$. | 21. $5A + 16K$. | 35. $2p^3 - q^6$. |
| 11. $3C + 8D$. | 22. $x^2 + 1$. | 36. $1 - 8a^8$. |
| | 23. $a^2 - 3$. | 37. $B^2 - 4AC$. |
| | 24. $n^2 + 5$. | 38. $x^2y^2 + 5$. |
| | 25. $t^2 - 10$. | 39. $A^2 - 4A$. |



40. Show geometrically, by the accompanying diagram, that

$$(x + y)^2 = x^2 + 2xy + y^2.$$

41. Show geometrically, by this diagram, that

$$(x - y)^2 = x^2 - 2xy + y^2.$$

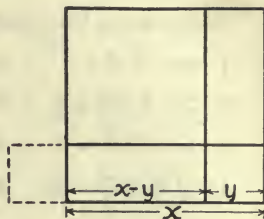
Solve:

42. $(t + 4)^2 = t(t + 3) + 36.$

43. $(2V + 3)^2 = (2V + 1)(2V + 3).$

44. $(4k - 7)^2 = (4k - 5)(4k + 3).$

45. $(u + 2)^2 - u^2 = u + 5.$



79. Squares of Arithmetical Numbers.—By the method of § 78, the squares of many arithmetical numbers are easily found without the aid of pencil.

Thus, $62^2 = (60 + 2)^2 = 3600 + 240 + 4 = 3844,$
and $79^2 = (80 - 1)^2 = 6400 - 160 + 1 = 6241.$

EXERCISES

Find mentally the squares of:

- | | | | | |
|--------|--------|----------|----------|----------|
| 1. 41. | 5. 68. | 9. 92. | 13. 109. | 17. 249. |
| 2. 59. | 6. 89. | 10. 99. | 14. 199. | 18. 301. |
| 3. 72. | 7. 78. | 11. 98. | 15. 148. | 19. 502. |
| 4. 38. | 8. 83. | 12. 101. | 16. 498. | 20. 999. |

80. Square of a Polynomial.—By actual multiplication, it is found that:

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

In general,

The square of any polynomial equals the sum of the squares of all of its terms, plus two times the algebraic product of each term into all of the terms following it.

EXAMPLE.— $(2n^2 - 3n + 5)^2 = (2n^2)^2 + (-3n)^2 + (5)^2 + 2(2n^2)(-3n)$
 $+ 2(2n^2)(5) + 2(-3n)(5)$
 $= 4n^4 + 9n^2 + 25 - 12n^3 + 20n^2 - 30n$
 $= 4n^4 - 12n^3 + 29n^2 - 30n + 25.$

EXERCISES

Write down the squares of the following polynomials:

1. $a - b + c$.
2. $2x + 3y + z$.
3. $m - 2n - p$.
4. $A + 3B - C$.
5. $2u - 6v + w$.
6. $A^2 + 2A + 1$.
7. $x^2 - 2x + 3$.
8. $r^2 + s^2 - t^2$.
9. $a + b + c + d$.
10. $x - y + z - w$.
11. $a^3 - a^2 - a + 1$.
12. $1 + 2P - 3P^2 + 4P^3$.

81. Quotients of the Form $a^3 - b^3$ Divided by $a - b$.—By actual division,

$$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2.$$

That is,

If the difference between the cubes of two terms is divided by the difference between those terms, the quotient equals the square of the first term, plus the product of the two terms, plus the square of the second term.

$$\begin{aligned} \text{EXAMPLE.} - \frac{8a^3 - 27b^3}{2a - 3b} &= (2a)^2 + (2a)(3b) + (3b)^2 \\ &= 4a^2 + 6ab + 9b^2. \end{aligned}$$

EXERCISES

Find mentally the values of the indicated quotients:

1. $\frac{x^3 - y^3}{x - y}$.
2. $\frac{m^3 - n^3}{m - n}$.
3. $\frac{w^3 - v^3}{w - v}$.
4. $\frac{x^3 - 1}{x - 1}$.
5. $\frac{1 - P^3}{1 - P}$.
6. $\frac{D^3 - 8}{D - 2}$.
7. $\frac{s^3 - 27}{s - 3}$.
8. $\frac{B^3 - 64}{B - 4}$.
9. $\frac{125 - y^3}{5 - y}$.
10. $\frac{a^3 - 8b^3}{a - 2b}$.
11. $\frac{27t^3 - 1}{3t - 1}$.
12. $\frac{d^3 - 64s^3}{d - 4s}$.
13. $\frac{8w^3 - 27v^3}{2w - 3v}$.
14. $\frac{125M^3 - N^3}{5M - N}$.
15. $\frac{64A^3 - 125B^3}{4A - 5B}$.

- | | | |
|---------------------------------------|------------------------------------|---------------------------------------|
| 16. $\frac{1000r^3 - 1}{10r - 1}.$ | 20. $\frac{8 - a^3b^3}{2 - ab}.$ | 24. $\frac{8a^3 - 27b^6}{2a - 3b^2}.$ |
| 17. $\frac{27b^3 - 216c^3}{3b - 6c}.$ | 21. $\frac{a^6 - 1}{a^2 - 1}.$ | 25. $\frac{64W^6 - g^3}{4W^2 - g}.$ |
| 18. $\frac{64H^3 - 343D^3}{4H - 7D}.$ | 22. $\frac{m^6 - n^6}{m^2 - n^2}.$ | 26. $\frac{M^9 - N^9}{M^3 - N^3}.$ |
| 19. $\frac{x^3y^3 - 1}{xy - 1}.$ | 23. $\frac{1 - t^9}{1 - t^3}.$ | 27. $\frac{1 - 8y^{12}}{1 - 2y^4}.$ |

Find mentally the following indicated products:

28. $(a - b)(a^2 + ab + b^2).$
 29. $(m - n)(m^2 + mn + n^2).$
 30. $(W - V)(W^2 + WV + V^2).$
 31. $(x - 1)(x^2 + x + 1).$
 32. $(2a - 1)(4a^2 + 2a + 1).$
 33. $(3x - 2y)(9x^2 + 6xy + 4y^2).$
 34. $(4s - 5t)(16s^2 + 20st + 25t^2).$
 35. $(x^2 - 1)(x^4 + x^2 + 1).$
 36. $(1 - 2D^2)(1 + 2D^2 + 4D^4).$

82. Quotients of the Form $a^3 + b^3$ Divided by $a + b$. — By actual division,

$$(a^3 + b^3) \div (a + b) = a^2 - ab + b^2.$$

That is,

If the sum of the cubes of two terms is divided by the sum of these terms, the quotient equals the square of the first term, minus the product of the two terms, plus the square of the second term.

EXAMPLE. — $\frac{125S^3 + 8t^3}{5S + 2t} = (5S)^2 - (5S)(2t) + (2t)^2$
 $= 25S^2 - 10St + 4t^2.$

EXERCISES

Find mentally the values of the indicated quotients:

1. $\frac{x^3 + y^3}{x + y}$.

10. $\frac{w^3 + 27 v^3}{w + 3 v}$.

19. $\frac{x^6 + 1}{x^2 + 1}$.

2. $\frac{W^3 + g^3}{W + g}$.

11. $\frac{P^3 + 125 Q^3}{P + 5 Q}$.

20. $\frac{1 + a^9}{1 + a^3}$.

3. $\frac{m^3 + n^3}{m + n}$.

12. $\frac{8 m^3 + n^3}{2 m + n}$.

21. $\frac{W^6 + B^3}{W^2 + B}$.

4. $\frac{a^3 + 1}{a + 1}$.

13. $\frac{27 D^3 + t^3}{3 D + t}$.

22. $\frac{8 y^6 + 27 z^6}{2 y^2 + 3 z^2}$.

5. $\frac{y^3 + 8}{y + 2}$.

14. $\frac{64 x^3 + 125 y^3}{4 x + 5 y}$.

23. $\frac{125 P^6 + 1}{5 P^2 + 1}$.

6. $\frac{27 + t^3}{3 + t}$.

15. $\frac{1000 K^3 + a^3}{10 K + a}$.

24. $\frac{216 + a^3 t^6}{6 + a t^2}$.

7. $\frac{64 + N^3}{4 + N}$.

16. $\frac{216 m^3 + 27 n^3}{6 m + 3 n}$.

25. $\frac{343 + 8 v^9 t^6}{7 + 2 v^3 t^2}$.

8. $\frac{x^3 + 125}{x + 5}$.

17. $\frac{b^3 + 343 d^3}{b + 7 d}$.

26. $\frac{H^9 + K^9}{H^3 + K^3}$.

9. $\frac{1 + 8 b^3}{1 + 2 b}$.

18. $\frac{8 R^3 + 125 c^3}{2 R + 5 c}$.

27. $\frac{27 + u^{12}}{3 + u^4}$.

Find mentally the following indicated products:

28. $(m + n)(m^2 - mn + n^2)$.

33. $(4 S + 1)(16 S^2 - 4 S + 1)$.

29. $(x + 1)(x^2 - x + 1)$.

34. $(a + 5 b)(a^2 - 5 ab + 25 b^2)$.

30. $(1 + a)(1 - a + a^2)$.

35. $(1 + x^2)(1 - x^2 + x^4)$.

31. $(W + V)(W^2 - WV + V^2)$.

36. $(y^2 + z^3)(y^4 - y^2 z^3 + z^6)$.

32. $(3 P + 1)(9 P^2 - 3 P + 1)$.

37. $(2 + 3 A)(4 - 6 A + 9 A^2)$.

SUPPLEMENTARY EXERCISES

Give the squares of:

1. a^n ; S^{2n} ; $-x^{3n}$; $A^n B^n$; y^{n+1} ; $-t^{n-1}$; a^{2n+3} .
2. $3X^a$; $5X^{a+1}Y^{a-1}$; $-12a^x b^{3x}$; $-7V^{2r-1}W^{2r+1}$.

Give the square roots of:

3. X^{2n} ; Y^{6a} ; $P^{4n}Q^{10n}$; t^{2n+2} ; v^{4x+6} .
4. $16D^{6a+8}$; $25X^{8n-2}Y^{8n+2}$; $81a^{4n}t^{2n+4}$.

Give the cubes of:

5. X^n ; a^{2x} ; $-P^{3v}$; $W^n V^n$; t^{n+1} ; t^{n-1} ; $-y^{3n+2}$.
6. $2A^n$; $3x^{v+1}y^{v-1}$; $-4M^{4a}N^{2a}$; $-5c^{2v-1}d^{2v+1}$.

Give the cube roots of:

7. w^{3n} ; Q^{6t} ; r_1^{12p} ; $-A^{6d}$; $-8x^{3n}y^{6n}$.
8. $27x^{6n}t^{9m}$; $-64N^3g^{12t}$; $-125a^{9n}t^{18n}$.

Give at sight the products of:

- | | |
|------------------------------------|------------------------------------|
| 9. $(x^n + 1)(x^n - 1)$. | 13. $(A^n + 3)(A^n + 7)$ |
| 10. $(V^t + 3)(V^t - 3)$. | 14. $(D^{2t} + 6)(D^{2t} - 4)$. |
| 11. $(a^{n+1} + 2)(a^{n+1} - 2)$. | 15. $(N^x - 1)(N^x + 8)$. |
| 12. $(x^a + y^b)(x^a - y^b)$. | 16. $(3y^{2n} + 6)(3y^{2n} - 2)$. |

Give the squares of:

- | | | |
|-------------------|---------------------------|--------------------------|
| 17. $a^n + 1$. | 19. $V^t + T^v$. | 21. $5H^{4t} - d^{4t}$. |
| 18. $x^n - y^n$. | 20. $2y^{3a} + 3w^{2a}$. | 22. $3p^{2d} + q^{2c}$. |

The products of polynomials often may be obtained mentally by grouping their terms so as to form binomials.

Find the products of the following by writing the factors as binomials, and using the rule in § 74.

$$23. (a + b + c)(a + b - c).$$

$$\begin{aligned}\text{SOLUTION.}-(a + b + c)(a + b - c) &= ([a + b] + c)([a + b] - c) \\ &= [a + b]^2 - c^2 \\ &= a^2 + 2ab + b^2 - c^2.\end{aligned}$$

$$24. (x + y - 1)(x + y + 1).$$

$$27. (b - c + 6)(b - c - 6).$$

$$25. (a^2 + a + 1)(a^2 + a - 1).$$

$$28. (S + 2t - 3)(S + 2t + 3).$$

$$26. (m - n + p)(m - n - p).$$

$$29. (a - 5b - c)(a - 5b + c).$$

$$30. (N + 3v - 5w)(N + 3v + 5w).$$

$$31. (a - 2b + c)(a + 2b + c).$$

SUGGESTION.—Group a and c .

$$32. (P - Q + R)(P + Q + R).$$

$$33. (x^2 + x + 1)(x^2 - x + 1).$$

$$34. (a^2 - ab + b^2)(a^2 + ab + b^2).$$

$$35. (a + b - c)(a - b + c).$$

SOLUTION.—Since b and c both have different signs in the two factors, group them. Then

$$\begin{aligned}(a + b - c)(a - b + c) &= (a + [b - c])(a - [b - c]) \\ &= a^2 - [b - c]^2 \\ &= a^2 - [b^2 - 2bc + c^2] \\ &= a^2 - b^2 + 2bc - c^2.\end{aligned}$$

$$36. (4n - 3x - y)(4n + 3x + y).$$

$$37. (2u + 3v - w)(2u - 3v + w).$$

$$38. (3P - 4T + V)(3P + 4T - V).$$

$$39. (a + b + c + d)(a + b - c - d).$$

SUGGESTION.—Group a and b into one term, and c and d into one term. Thus, $([a + b] + [c + d])([a + b] - [c + d])$.

$$40. (x - 2y - t + z)(x - 2y + t - z).$$

$$41. (R + 2S - 2T + U)(R + 2S + 2T - U).$$

$$42. (x^3 - x^2 - x + 1)(x^3 - x^2 + x - 1).$$

$$43. (1 + a - a^2 - a^3)(1 + a + a^2 + a^3).$$

Find the products of the following by writing the factors as binomials and proceeding as in § 76:

44. $(a + b + 3)(a + b - 5)$.

SOLUTION. — $(a + b + 3)(a + b - 5) = ([a + b] + 3)([a + b] - 5)$
 $= [a + b]^2 - 2[a + b] - 15$
 $= a^2 + 2ab + b^2 - 2a - 2b - 15.$

45. $(x - y + 2)(x - y + 3)$.

46. $(p + q + 10)(p + q - 16)$.

47. $(2W - 3V + 8)(2W - 3V - 3)$.

48. $(x^2 + x + 6)(x^2 + x - 2)$.

49. $(t^2 - t + 10)(t^2 - t - 14)$.

50. $(n^2 - 1 + n)(n^2 - 2 + n)$.

CHAPTER VIII

FACTORS. MULTIPLES. EQUATIONS SOLVED BY FACTORING

83. Factoring.—What are the factors of 6? Of 10? Of 12? Of mn ? Of $v(v+t)$?

The process of finding the factors of a given expression is called **factoring**.

If a factor of a given expression does not itself have factors, it is called a **prime factor** of the expression. In general, *to factor an expression means to find its prime factors.*

Facility in the factoring of many expressions depends upon an ability to recognize the special forms of the expressions and upon a knowledge of the rules in Chapter VII.

84. Polynomial with a Monomial Factor.—If each term of a given polynomial is divisible by the same monomial, the polynomial is divisible by the monomial, and hence the monomial is a factor.

For example, in $18a^4 - 9a^3 + 27a^2$, each term is divisible by $9a^2$.

$$\begin{array}{r} 2a^2 - a + 3 \\ 9a^2 \overline{) 18a^4 - 9a^3 + 27a^2} \end{array}$$

Hence, since the dividend always equals the product of the divisor and quotient,

$$18a^4 - 9a^3 + 27a^2 = 9a^2(2a^2 - a + 3).$$

That is, the factors of $18a^4 - 9a^3 + 27a^2$ are $9a^2$ and $2a^2 - a + 3$.

Hence the rule for finding the factors of a polynomial with a monomial factor:

Find, by inspection, the monomial of highest power that will divide each term of the polynomial. Divide the polynomial by this monomial. The divisor and quotient are the monomial and polynomial factors, respectively, of the given polynomial.

The division should always be performed *mentally*, and the work written as shown in the following example:

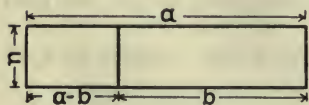
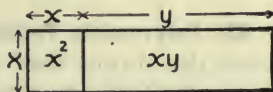
EXAMPLE.—Factor $2nx^2 - 4ny^2 + 6nz^2$.

$$2nx^2 - 4ny^2 + 6nz^2 = 2n(x^2 - 2y^2 + 3z^2).$$

EXERCISES

Factor:

1. $2a + 2b$.
2. $3x - 6y$.
3. $4B^2 - 16AC$.
4. $5nR + 10nQ$.
5. $8dx^2 - 3dy^2$.
6. $A^3 - 4A^2$.
7. $\pi R^2 + \pi R'^2$.
8. $x^6 + 5x^4$.
9. $n^2m^2 - m^4$.
10. $2\pi R^2 + 2\pi RH$.
11. $s^6t^2 + s^2t^6$.
12. $5w^3 - 10w^2v$.
13. $18x^5 - 9x^3$.
14. $c^4n + 2c^2n^2$.
15. $7P^4 + 21P^3$.
16. $x^3 - x^2 + x$.
17. $4V_1^2 + 6V_1V_2 + 4V_1V_3$.
18. $H^6 - 6H^5b + 2H^4b^2$.
19. $v^2t + 2vt^2 + vt^3$.
20. $8n^3m^3 + 6n^2m^4 + 4nm^5$.
21. $3a^4b - 6a^3b^2 + 3a^2b^3$.
22. $2\pi R^2 + 2\pi r^2 + 2\pi Rr$.
23. $a^4b^5c^2 + a^2b^4c^6 + a^5b^2c^4$.
24. $24w^5d^3 - 12w^3d + 42w^6d^2$.
25. $6v^8 + 8v^{10} - 16v^6$.
26. $-x^3 - x^2 - x$.
27. $-a^2 - ab - b^2$.
28. $4x^6 - 6x^4y^2 + 8x^3y^3 - 12x^2y^4$.
29. $-10M^8 + 12M^6 - 16M^4 + 8M^2$.
30. $56a^3y^5 - 14a^4y^4 + 28a^5y^3 - 105a^6y^2$.
31. Show by this diagram that $x^2 + xy = x(x + y)$.



32. Show by a diagram that

$$na - nb = n(a - b).$$

33. Many arithmetical computations may be shortened by applying

the process of factoring in § 84. Thus,

$$4 \times 7 + 6 \times 7 = 10 \times 7, \text{ or } 70.$$

Find by this method the values of the following: $17 \times 45 + 13 \times 45$; $38 \times 92 + 42 \times 92$; $64 \times 575 + 85 \times 575 - 24 \times 575$; $9 \times 3.1416 + 16 \times 3.1416$; $2 \times 16 \times 8 + 2 \times 24 \times 8 + 16 \times 24$; $365 \times 784 - 15 \times 784 + 50 \times 784$.

34. If, in two numbers of two digits each, the tens' digits are equal and the sum of the ones' digits is 10, the numbers may be expressed by $10t + u$ and $10t + 10 - u$, respectively, in which t is the tens' digit, u the ones' digit of one, and $10 - u$ the ones' digit of the other.

Show that their product is $100t^2 + 100t + 10u - u^2$.

Show that this may be written $100t(t + 1) + u(10 - u)$.

Hence, *to multiply two numbers of which the tens' digits are equal, and the sum of the ones' digits is 10, take the product of the number of tens by a number one greater for the hundreds and annex the product of the ones.*

Thus, $74 \times 76 = 5624$; $69 \times 61 = 4209$; $43 \times 47 = 2021$.

35. Give at sight the products of the following: 23×27 ; 16×14 ; 26×24 ; 32×38 ; 46×44 ; 56×54 ; 68×62 ; 77×73 ; 88×82 ; 93×97 ; 81×89 ; 84×86 ; 79×71 ; 75×75 ; 91×99 ; 94×96 .

36. State a rule for squaring any number ending in 5 that follows as a special case of the rule in Exercise 34.

37. Give at sight the squares of the following:

25; 45; 35; 65; 85; 55; 95; 75; 105; 115.

85. Polynomials Factored by Grouping Terms.—Some polynomials that do not have monomial factors may be factored by the method of § 84, after first grouping the terms.

EXAMPLE 1. — Factor $a^2 + ab + ac + bc$.

Grouping the first two terms and factoring them, then the last two, gives $a(a + b) + c(a + b)$.

Now the factor $a + b$ is common to the two terms. Dividing by $a + b$ gives a quotient $a + c$.

Hence the factors are $a + b$ and $a + c$.

That is, $a^2 + ab + ac + bc = (a + b)(a + c)$.

EXAMPLE 2. — Factor $mt + nv - nt - mv$.

Grouping the first and fourth terms, and the second and third terms,

$$\begin{aligned} mt + nv - nt - mv &= m(t - v) - n(t - v) \\ &= (t - v)(m - n). \end{aligned}$$

Notice that we factor out $-n$ from the second group, rather than $+n$, so that the expressions left in parentheses will be the same.

These examples illustrate the rule :

Group the terms of the polynomial to be factored so that a monomial factor may be divided out from each group, and that the expressions left in parentheses are the same in all of the groups. Then divide by the expression in parentheses, writing the divisor as one factor and the quotient as the other.

If in Example 2 we should group the second and fourth terms, factoring out v , and group the first and third terms, factoring out t , we would get the factors $(v-t)(n-m)$. It is evident that these factors differ from those in the solution above only in their signs. So in other types of problems in factoring, *two sets of factors* may be obtained, differing only in sign. For practical purposes only one set of factors is ever needed, and the other may be ignored.

NOTE. — The student should check his work in every problem in factoring by some method. In most cases, this is easily accomplished by multiplying mentally the factors obtained, to see if the product equals the given expression to be factored. A second method, convenient in many cases, consists of assigning special values to all letters involved and comparing the values of the factors with the value of the given expression. Thus, if in Example 1 above we make $a = 1$, $b = 2$, $c = 3$, then the expression to be factored becomes 12 and the factors 3 and 4, respectively.

EXERCISES

Factor :

1. $3(x+y) + n(x+y)$.

2. $a(P-Q) + b(P-Q)$.

3. $v(a+b) - t(a+b)$.

4. $2x(m-n) - 3y(m-n)$.

5. $5w(h-k) + u(h-k)$.

6. $a(x-y) + b(y-x)$.

SUGGESTION. — This may be written $a(x-y) - b(x-y)$.

7. $s(A-5) + t(5-A)$.

8. $p(r-R) - q(R-r)$.

9. $B^2(N-M) + 4AC(M-N)$.

10. $x(w-v) + (v-w)$,

11. $ac + ad + bc + bd$.

12. $nm + ny + mx + xy$.

13. $vt + wt + 2v + 2w$.

14. $Wg - Wh + gk - hk$.

15. $mp + np - mq - nq$.

16. $as - bs - at + bt$.

17. $vu - vR + uS - RS$.

18. $mn + mt - ns - st$.

19. $x^2 + xy + xz + yz.$

20. $p^2 + pr - pq - qr.$

21. $M^2 - MN + MP - NP.$

22. $2v + v^2 - 2u - vu.$

23. $3rs - 2sx + 3ry - 2xy.$

24. $2t - tm - 2m + 4.$

25. $h^2k^2 - m^2k^2 - h^2 + m^2.$

26. $W^2x - V^2x - W^2y + V^2y.$

27. $FC + GC + 2F + 2G.$

28. $pq - 3p - 2q + 6.$

29. $R^2 - R - Rr + r.$

30. $2a^2 - cx^2 - a^2c + 2x^2.$

86. Trinomials that are Perfect Squares.—If a binomial is squared, as in § 78, the result is always a trinomial. A trinomial which is the square of a binomial is called a **perfect square**.

Since any binomial must be either of the form $a + b$ or $a - b$, and since $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$, evidently any trinomial which is a perfect square must be either of the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$. In either case, the term $2ab$ is twice the product of the absolute values of $\sqrt{a^2}$ and $\sqrt{b^2}$. Hence,

A trinomial is a perfect square if one term is twice the product of the absolute values of the square roots of the other two terms.

In $9x^2 - 24xy + 16y^2$, what are the square roots of $9x^2$ and $16y^2$? What is twice the product of their absolute values? Is the trinomial a perfect square?

Since the binomial factors of a trinomial which is a perfect square are equal, *to factor such a trinomial is equivalent to expressing it as the square of a binomial.*

EXAMPLE 1.—Factor $25m^2 - 40m + 16$.

$$\sqrt{25m^2} = \pm 5m, \text{ and } \sqrt{16} = \pm 4.$$

Since twice the product of these roots must give $-40m$, the positive root of one term and the negative root of the other must be chosen, which can be done in two ways.

Hence, $25m^2 - 40m + 16 = (5m - 4)^2$ or $(-5m + 4)^2$.

Check.—When $m = 2$, the trinomial is 36 and the factors 6 and 6, or -6 and -6 .

EXAMPLE 2.—Factor $64v^2 + 112vt + 49t^2$.

$$\sqrt{64v^2} = \pm 8v, \text{ and } \sqrt{49t^2} = \pm 7t.$$

Since twice the product of these roots must give $+112vt$, they must be taken with like signs, which can be done in two ways.

Hence, $64v^2 + 112vt + 49t^2 = (8v + 7t)^2$ or $(-8v - 7t)^2$.

It is seen here, as in § 85, that there are two sets of factors in each problem. In practice only one of these is used, viz. the set in which the first term is positive.

To find a binomial of which a given trinomial is a square, take as the terms the square roots of the two square terms of the trinomial with such signs that their product gives the sign of the remaining term.

EXERCISES

Determine which of the following are perfect squares:

- | | |
|---------------------------|-----------------------------|
| 1. $x^2 + 6x + 9$. | 9. $16H^2 - 12Ht + 9t^2$. |
| 2. $n^2 + 16 + 8n$. | 10. $49x^2 + 4y^2 - 20$. |
| 3. $1 - 4A + 4A^2$. | 11. $24W^4 - 10W^2 + 1$. |
| 4. $25t^2 + 12t + 1$. | 12. $a^4 - 2a^2b^2 - b^4$. |
| 5. $x^2 + ax + a^2$. | 13. $m^2 + 18ma + 81a^2$. |
| 6. $9v^2 - 6v + 4$. | 14. $b^2 - 12bc + 36c^2$. |
| 7. $M + 4MN + 4N^2$. | 15. $9p^2 + 38pq + 49q^2$. |
| 8. $r^2 - 10rs + 25s^2$. | 16. $D^4 + 1 + 2D^2$. |

Make perfect square trinomials of the following by supplying the missing terms:

- | | |
|--------------------------|------------------------------|
| 17. $x^2 + ? + y^2$. | 24. $36 + ? - 12x$. |
| 18. $a^2 - 2ab + ?$. | 25. $a^2 - ? + 100$. |
| 19. $? + 2mn + n^2$. | 26. $? + 1 + x^4$. |
| 20. $4A^2 + ? + 1$. | 27. $4P^3 + 1 + ?$. |
| 21. $16 - 24t + ?$. | 28. $r_1^2 - 12r_1r_2 + ?$. |
| 22. $25v^2 - ? + 9w^2$. | 29. $? - 32N + ?$. |
| 23. $W^2 + 49 + ?$. | 30. $? + 42b^2 + ?$. |

Factor :

31. $x^2 + 6x + 9.$

32. $R^2 - 4R + 4.$

33. $t^2 + 8t + 16.$

34. $25 - 10a + a^2.$

35. $1 + 12n + 36n^2.$

36. $49W^2 - 28W + 4.$

37. $v^2 + 2v + 1.$

38. $81 - 18A + A^2.$

39. $x^4 + 4x^2 + 4.$

40. $16 - 24D^2 + 9D^4.$

41. $ax^2 - 6axy + 9ay^2.$

(First remove monomial factor a .)

42. $2n^2 + 20nm + 50m^2.$

43. $tS^2 - 30tSV + 225tV^2.$

44. $16H^2 + 56Hd + 49d^2.$

45. $25b^2 - 60bc + 36c^2.$

46. $4w^2v + 36wv^2 + 81v^3.$

47. $M^2 - 50MN + 625N^2.$

48. $9r_1^2 + 30r_1r_2 + 25r_2^2.$

49. $x^5 - 2x^3y^2 + xy^4.$

50. $a^4 + 4a^3b^2 + 4a^2b^4.$

51. $P^2 - 6PQ^3 + 9Q^6.$

52. $81m^4 + 36m^2n + 4n^2.$

53. $25W^4 - 40W^2g + 16g^2.$

54. $x^3y^2 + 2x^2y + x.$

55. $c^2d^2 + 2cdmn + m^2n^2.$

56. $R^2r^2 - 8Rrt + 16t^2.$

57. $24v^2 + 36v^3 + 4v.$

58. $25z^2 + 81z^4 + 90z^3.$

59. $64t_1^2 + 49t_2^2 - 112t_1t_2.$

60. $36A^3 + 144A + 144A^2.$

61. $-1 - 6a - 9a^2.$

87. Trinomials of the Form $x^2 + ax + b$.—By the principle in § 76, the product of two binomials having a common term x is always a trinomial in the general form $x^2 + ax + b$.

Thus, $(x + 5)(x + 7) = x^2 + 12x + 35$.

Here a equals 12 and b equals 35.

It follows that a trinomial in the form $x^2 + ax + b$ may be factored into two binomials with a common term x , if for the other terms of the factors two numbers can be found whose algebraic sum is the coefficient a and whose algebraic product is b .

EXAMPLE 1.—Factor $x^2 - 8x + 15$.

For the second terms of the factors, we are to find two numbers whose sum is -8 and whose product is $+15$. These are evidently -3 and -5 .

Hence, $x^2 - 8x + 15 = (x - 3)(x - 5)$.

Check.—When $x = 1$, the trinomial is 8, and the factors -2 and -4 , respectively; or mentally multiplying, $(x-3)(x-5) = x^2 - 8x + 15$.

EXAMPLE 2.—Factor $m^2 - 4mn - 45n^2$.

We are to find two numbers whose algebraic sum is $-4n$ and product $-45n^2$. These are evidently $-9n$ and $+5n$.

Hence, $m^2 - 4mn - 45n^2 = (m - 9n)(m + 5n)$.

EXERCISES

Factor:

- | | | |
|------------------------|------------------------|---------------------------|
| 1. $t^2 + 5t + 6$. | 20. $p^2 + 14p + 48$. | 39. $x^2 + 2x - 3$. |
| 2. $a^2 + 5a + 4$. | 21. $B^2 - 5B + 6$. | 40. $R^2 + 3R - 10$. |
| 3. $n^2 + 8n + 12$. | 22. $N^2 - 5N + 4$. | 41. $K^2 + 3K - 4$. |
| 4. $P^2 + 9P + 14$. | 23. $w^2 - 8w + 12$. | 42. $s^2 - 35 + 2s$. |
| 5. $x^2 + 9x + 20$. | 24. $Q^2 - 9Q + 14$. | 43. $y^2 + 3y - 18$. |
| 6. $S^2 + 9S + 8$. | 25. $z^2 - 9z + 20$. | 44. $3a - 28 + a^2$. |
| 7. $v^2 + 10v + 16$. | 26. $t^2 - 10t + 9$. | 45. $C^2 - 12 + 4C$. |
| 8. $a^2 + 10a + 9$. | 27. $D^2 - 10D + 16$. | 46. $5m + m^2 - 14$. |
| 9. $R^2 + 10R + 24$. | 28. $V^2 - 10V + 21$. | 47. $b^2 + 3b - 40$. |
| 10. $k^2 + 10k + 21$. | 29. $a^2 - 10a + 24$. | 48. $5x - 36 + x^2$. |
| 11. $V^2 + 11V + 18$. | 30. $T^2 - 9T + 8$. | 49. $B^2 - 3B - 18$. |
| 12. $d^2 + 11d + 24$. | 31. $y^2 - 17y + 30$. | 50. $g^2 - 3g - 40$. |
| 13. $M^2 + 11M + 30$. | 32. $H^2 - 14H + 45$. | 51. $m^2 - 14 - 5m$. |
| 14. $b^2 + 12b + 20$. | 33. $W^2 - 4W + 3$. | 52. $T^2 - 4T - 12$. |
| 15. $y^2 + 12y + 27$. | 34. $m^2 - 12m + 32$. | 53. $x^2 + 5xy + 6y^2$. |
| 16. $A^2 + 12A + 32$. | 35. $c^2 + 35 - 12c$. | 54. $a^2 + 7ab + 10b^2$. |
| 17. $r^2 + 15r + 54$. | 36. $B^2 + 55 - 16B$. | 55. $w^2 - 5wv + 6v^2$. |
| 18. $m^2 + 17m + 70$. | 37. $t^2 + 7 - 8t$. | 56. $r^2 - 7rs + 10s^2$. |
| 19. $c^2 + 13c + 30$. | 38. $F^2 + 2F - 15$. | 57. $p^2 - 3pq - 28q^2$. |

58. $C^2 - 5 CD - 14 D^2$.

59. $y^2 + 5 yz - 14 z^2$.

60. $a^2b^2 - 3 ab - 18$.

61. $m^2n^2 - 2 mn - 35$.

62. $H^2 - 19 H - 120$.

63. $V^2 - V - 30$.

64. $ab^2 - 3 ab - 70 a$.

(First remove monomial factor a .)

65. $xy^2 + xy - 90 x$.

66. $w^2v^2 + 2 w^2v - 99 w^2$.

67. $d^3m^2 + d^3m - 56 d^3$.

68. $2 st^2 - 56 st + 230 s$.

69. $3 x^2 - 90 x + 600$.

70. $6 b^2c + 126 bc + 660 c$.

71. $81 t^2 + 30 at^2 + a^2t^2$.

72. $4 wn^2 - 28 wn - 240 w$.

73. $90 vy - 204 v + 6 vy^2$.

74. $6 - 5 a - a^2$.

88. Trinomials of the Form $ax^2 + bx + c$.—There are different methods of factoring trinomials of the form $ax^2 + bx + c$. Two methods are given here.

FIRST METHOD.—If any two binomials such as $mx + p$ and $nx + q$ are multiplied, their product equals $mnx^2 + mqx + np x + pq$, which may be written $mnx^2 + (mq + np)x + pq$. This is of the general form $ax^2 + bx + c$.

Thus, $(2x + 3)(3x + 1) = 6x^2 + 11x + 3$.

If a trinomial of the general form $ax^2 + bx + c$ has binomial factors of the forms $mx + p$ and $nx + q$, these may be found by trial.

EXAMPLE 1.—Factor $2x^2 + 5x + 2$. The first terms of the binomial factors must be factors of $2x^2$, and the second terms must be factors of 2. The sum of the products of the first term of each by the second of the other, called the *cross products*, must be $5x$. Since all signs of the trinomial are positive, only positive terms for the binomial factors need be sought.

The possible trial sets of factors may be arranged as follows:

$$\begin{array}{r} 2x + 2 \\ \underline{x + 1} \end{array} \qquad \begin{array}{r} x + 2 \\ \underline{2x + 1} \end{array}$$

It is seen that the second of these gives the right sum of cross products.

Hence, the factors are $x + 2$ and $2x + 1$.

Check.—When $x = 1$, the trinomial is 9, and the factors 3 and 3, respectively; or multiplying the factors, $(x + 2)(2x + 1) = 2x^2 + 5x + 2$.

EXAMPLE 2.—Factor $3w^2 + ww - 10v^2$.

Since the third term is negative, the factors of it chosen must have unlike signs.

Four of the many possible trial sets of factors are:

$w - 2v$	$w + 2v$	$w - 5v$	$w + 5v$
<u>$3w + 5v$</u>	<u>$3w - 5v$</u>	<u>$3w + 2v$</u>	<u>$3w - 2v$</u>

It is seen that the second of these gives the right sum of cross products.

Hence, $3w^2 + wv - 10v^2 = (w + 2v)(3w - 5v)$.

Each trial set should be tested as written. If done, it usually will be unnecessary to write down all possible sets. Each set should be written down as above, and the sum of cross products found mentally. In a short time the student should be able to factor most trinomials in very few trials. The following principles systematize the work:

I. *If the trinomial contains no monomial factor, neither factor can contain one.*

Thus, in Example 1, since $2x^2 + 5x + 2$ contains no monomial factor, it was useless to write the first trial set, with the binomial $2x + 2$.

II. *If the last term of the trinomial is +, the last terms of the factors will be both + or both -, according as the middle term of the trinomial is + or -.*

III. *If the last term of the trinomial is -, the last terms of the factors will have unlike signs.*

SECOND METHOD. — In $mnx^2 + mqx + np$, which is the product of $mx + p$ and $nx + q$, the product of the terms mnx^2 and pq equals the product of the terms mqx and np . From this we may infer that

If we may separate the second term of a trinomial in the form $ax^2 + bx + c$ into two terms whose product is equal to that of the first and third terms, we may factor the expression as in § 85.

EXAMPLE 3. — Factor $3x^2 + 11x + 6$.

$3x^2 \cdot 6 = 18x^2$. Now $18x^2 = 9x \cdot 2x = 18x \cdot x$. The sum of the factors $9x$ and $2x$ is the middle term. Hence we write

$$\begin{aligned} 3x^2 + 11x + 6 &= 3x^2 + 9x + 2x + 6 \\ &= 3x(x + 3) + 2(x + 3) \\ &= (x + 3)(3x + 2). \end{aligned}$$

Note that it will be necessary only to use the coefficients in determining how to break up the middle term, as shown in the next example.

EXAMPLE 4. — Factor $12t^2 - 5tv - 2v^2$.

$12(-2) = -24$. $24 = 3 \times 8 = 6 \times 4 = \text{etc.}$ Since the product -24 has a negative sign, the two terms into which $-5tv$ is to be separated must have unlike signs. The required terms are $-8tv$ and $3tv$.

$$\begin{aligned}\text{Hence, } 12t^2 - 5tv - 2v^2 &= 12t^2 - 8tv + 3tv - 2v^2 \\ &= 4t(3t - 2v) + v(3t - 2v) \\ &= (3t - 2v)(4t + v).\end{aligned}$$

EXERCISES

Factor:

- | | |
|-------------------------|-------------------------------------|
| 1. $10a^2 + 17a + 7$. | 22. $4b^2 - 3b - 27$. |
| 2. $3t^2 + 11t + 10$. | 23. $6n^2 - 17n - 14$. |
| 3. $3x^2 + 10x + 3$. | 24. $3L^2 - 2L - 5$. |
| 4. $5m^2 + 12m + 4$. | 25. $3F^2 - F - 4$. |
| 5. $2V^2 + 9V + 10$. | 26. $3R^2 - R - 2$. |
| 6. $3A^2 + 10A + 7$. | 27. $2a^2x - ax - 28x$. |
| 7. $8r^2 + 22r + 15$. | (First remove the monomial factor.) |
| 8. $3B^2 - 11B + 10$. | |
| 9. $10y^2 - 17y + 7$. | 28. $6c^2d - cd - 35d$. |
| 10. $3S^2 - 10S + 3$. | 29. $6z^2 - 7z - 20$. |
| 11. $5g^2 - 12g + 4$. | 30. $20x^2 - 46x - 84$. |
| 12. $15 - 22x + 8x^2$. | 31. $36g^2 - 15g - 6$. |
| 13. $2P^2 - 9P + 10$. | 32. $4ay^2 - 10ay - 36a$. |
| 14. $5 - 11c + 2c^2$. | 33. $15W^2g - 5Wg - 20g$. |
| 15. $4 - 8k + 3k^2$. | 34. $30w^2v^2 - 37wv^2 - 7v^2$. |
| 16. $4R^2 + 3R - 27$. | 35. $2Q^2t - 5Qt - 25t$. |
| 17. $6m^2 + 17m - 14$. | 36. $4C^2 + 32C + 15$. |
| 18. $3D^2 + 2D - 5$. | 37. $4x^2 + 23xy + 15y^2$. |
| 19. $3a^2 + a - 4$. | 38. $2M^2 + 15MN + 7N^2$. |
| 20. $3v^2 + v - 2$. | 39. $2d^2 - 3dk + k^2$. |
| 21. $4H^2 - H - 5$. | 40. $6E^2 - 23EF + 20F^2$. |

41. $14b^2 + 2c^2 - 11bc$. 46. $12G^2n^2 - 5Gdn^2 - 2d^2n^2$.
 42. $7A^2 + 4n^2 - 29An$. 47. $4b^2x - 2bdx - 20d^2x$.
 43. $21R^2 + 2Rt - 8t^2$. 48. $12Y^2 - 4YZ - 16Z^2$.
 44. $2r_1^2 - 5r_1r_2 - 18r_2^2$. 49. $3a^4 - a^2 - 2$.
 45. $3T^2S - 2TT'S - 5T'^2S$. 50. $6w^2t + 15wv^2t - 54v^4t$.

89. Binomials of the Form $a^2 - b^2$. Since the product of $a + b$ and $a - b$ is $a^2 - b^2$, the factors of $a^2 - b^2$ are $a + b$ and $a - b$. Hence,

The factors of the difference between the squares of two numbers are the sum and the difference of the numbers.

EXAMPLE.—Factor $4t^2 - 25$.

$$\begin{aligned} 4t^2 - 25 &= (2t)^2 - (5)^2 \\ &= (2t + 5)(2t - 5). \end{aligned}$$

Check.—When $t = 1$, the trinomial is -21 , and the factors 7 and -3 respectively. Or, check by finding the product of the factors.

Observe that, by taking the negative square roots of the terms, we might also write $4t^2 - 25 = (-2t)^2 - (-5)^2$

$$= (-2t - 5)(-2t + 5).$$

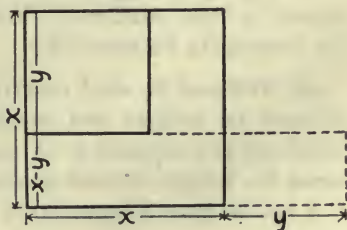
See § 84 and § 85.

EXERCISES

1. Show by the accompanying diagram that $x^2 - y^2 = (x + y)(x - y)$.

Factor:

2. $N^2 - 4$. 8. $k^2 - 25$.
 3. $a^2 - 9$. 9. $16 - S^2$.
 4. $V^2 - 1$. 10. $H^2 - 64$.
 5. $t^2 - 16$. 11. $49 - m^2$.
 6. $1 - P^2$. 12. $100 - v^2$.
 7. $4 - w^2$. 13. $x^2 - 81$.



14. $A^2 - 144$. 16. $1 - 49t^2$. 18. $h^2 - k^2$.
 15. $1 - 16x^2$. 17. $9 - 4B^2$. 19. $81m^2 - 64n^2$.

20. $x^2 - 49y^2$. 23. $36E^2 - F^2$. 26. $16x^2 - 121y^2$.
 21. $16R^2 - 25D^2$. 24. $81W^2 - 49t^2$. 27. $50a^2 - 32b^2$.
 22. $4c^2 - 25d^2$. 25. $C^2 - 100D^2$. (Remove the monomial factor first.)
 28. $x^2y^2 - y^2$. 33. $28V^2g^2 - 7g^4$.
 29. $w^3 - w$. 34. $x^2y^2 - 1$.
 30. $6at^2 - 24a$. 35. $4 - 36a^2t^2$.
 31. $16m^4 - 36m^2$. 36. $200P^2W^2 - 8W^4$.
 32. $H^2t - 25t^3$. 37. $m^4 - n^4$.



38. $16 - T^4$.

39. $a^4 - 1$.

40. $\pi r_1^2 - \pi r_2^2$.

41. $2\pi R^3 - 8\pi RH^2$.

42. Show that the area of the shaded border in this figure is $S = (A + a)(A - a)$.

43. Show that the area of the ring shown in the figure is $S = \pi(R + r)(R - r)$, where R is the outer radius and r the inner radius of the ring.

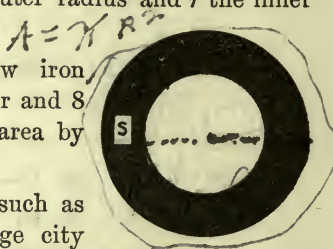
44. The cross section of a hollow iron column is 10 inches in outer diameter and 8 inches in inner diameter. Find the area by the formula in Problem 43.

45. The cost of steel construction such as is used in bridges and modern large city buildings is computed at so much per pound of steel used, and hence the weight of steel is always computed.

A hollow steel column is 16 ft. long, 14 in. in outer diameter, and 10 in. in inner diameter. Find its weight, allowing 490 lb. to a cubic foot. Use formula in Problem 43.

46. Find mentally the values of the following:

$$28^2 - 22^2; 37^2 - 33^2; 49^2 - 41^2; 76^2 - 74^2; 97^2 - 93^2; 116^2 - 114^2$$



90. Polynomials which can be Written in the Form $a^2 - b^2$. — Some polynomials, by grouping terms, can be written in the form $a^2 - b^2$, and hence factored by the method of § 89.

EXAMPLE 1. — Factor $w^2 + 2 wv + v^2 - t^2$.

Grouping terms,

$$\begin{aligned} w^2 + 2 wv + v^2 - t^2 &= (w^2 + 2 wv + v^2) - t^2 \\ &= (w + v)^2 - t^2 \\ &= (w + v + t)(w + v - t). \end{aligned}$$

Check. — When $w = 1$, $v = 1$, $t = 1$, the polynomial is 3, and the factors 3 and 1, respectively.

EXAMPLE 2. — Factor $A^2 - B^2 + 2 BC - C^2$.

$$\begin{aligned} A^2 - B^2 + 2 BC - C^2 &= A^2 - (B^2 - 2 BC + C^2) \\ &= A^2 - (B - C)^2 \\ &= (A + [B - C])(A - [B - C]) \\ &= (A + B - C)(A - B + C). \end{aligned}$$

Learn to omit all except the first and last steps, and to write out the work as follows :

EXAMPLE 3. — Factor $x^2 + 6x - m^2 + 4mn - 4n^2 + 9$.

$$\begin{aligned} x^2 + 6x - m^2 + 4mn - 4n^2 + 9 &= (x^2 + 6x + 9) - (m^2 - 4mn + 4n^2) \\ &= (x + 3 + m - 2n)(x + 3 - m + 2n). \end{aligned}$$

EXERCISES

Factor :

- | | |
|-----------------------------------|---|
| 1. $(a + b)^2 - 1$. | 10. $49 V^2 - (3 T - S)^2$. |
| 2. $(x - y)^2 - z^2$. | 11. $(x + y)^2 - (a + b)^2$. |
| 3. $(S + T)^2 - 25$. | 12. $(M - N)^2 - (W - V)^2$. |
| 4. $(h - k)^2 - 16 t^2$. | 13. $(C + D)^2 - (4 E - F)^2$. |
| 5. $(2 m + n)^2 - 36$. | 14. $(1 - x)^2 - (v - t)^2$. |
| 6. $1 - (A + B)^2$. | 15. $m^2 + 2 mn + n^2 - b^2$. |
| 7. $4 - (2 w + v)^2$. | 16. $R^2 - 4 RA + 4 A^2 - B^2$. |
| 8. $25 a^2 - (b - c)^2$. | 17. $1 + 2 x + x^2 - y^2$. |
| 9. $36 r_1^2 - (r_2 - 4 r_3)^2$. | 18. $r_1^2 + 6 r_1 r_2 + 9 r_2^2 - r_3^2$. |

19. $x^4 + 4x^3 + 4x^2 - 1.$

22. $1 - a^2 - 2ab - b^2.$

20. $W^2 + 9D^2 - 4M^2 - 6WD.$

23. $4 - M^2 + 6MV - 9V^2.$

21. $1 - 8x + 16x^2 - z^2.$

24. $1 - h^2 + 14hk - 49k^2.$

25. $A^2 - 9B^2 - 30BC - 25C^2.$

26. $16U^2 + 36VW - 4V^2 - 81W^2.$

27. $S^2 - T^2 + 4aT - 4a^2.$

28. $12pq + 16r^2 - 4p^2 - 9q^2.$

29. $A^2 - 4C^2 - 8AB + 16B^2.$

30. $6tm - t^2 - 9m^2 + 1.$

31. $a^2 + 2ab + b^2 - c^2 - 2cd - d^2.$

32. $x^2 - 4x + 4 - m^2 + 6mn - 9n^2.$

33. $1 + 2xy + 2p - y^2 - x^2 + p^2.$

34. $W^2 + 12at - 10Wg - 4a^2 + 25g^2 - 9t^2.$

35. $M^2 - 12dM - 4kN + 36d^2 - 4N^2 - k^2.$

36. $x^4 - E^2 + 10Em^2 - 16x^2n - 25m^4 + 64n^2.$

91. Binomials of the Form $a^3 - b^3$.—From § 81 it follows that the factors of $a^3 - b^3$ are $a - b$ and $a^2 + ab + b^2$. Test these factors by actual multiplication. Hence,

One factor of the difference between the cubes of two terms is the difference between the terms, and the other factor is the sum of their squares and their product.

EXAMPLE.—Factor $8m^3 - 125n^3$.

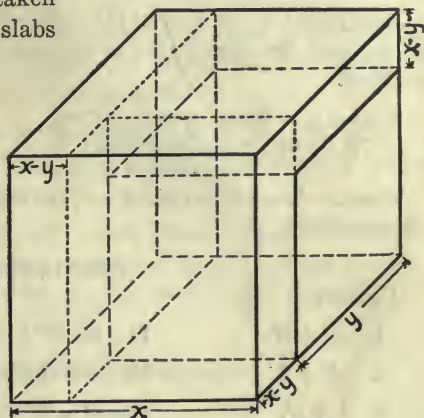
$$\begin{aligned} 8m^3 - 125n^3 &= (2m)^3 - (5n)^3 \\ &= (2m - 5n)(4m^2 + 10mn + 25n^2). \end{aligned}$$

Check.—When $m = 1$ and $n = 1$, the binomial is -117 , and the factors -3 and 39 , respectively.

EXERCISES

1. Show by the accompanying diagram that $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$. (If y^3 is taken from x^3 , it leaves three slabs each $x - y$ thick.)

Factor:



2. $m^3 - n^3$.

3. $x^3 - 8$.

4. $A^3 - 1$.

5. $1 - P^3$.

6. $8 - r^3$.

7. $27 - t^3$.

8. $8w^3 - v^3$.

9. $T^3 - 27U^3$.

10. $64y^3 - 1$.

11. $M^3 - 125N^3$.

16. $27x^4 - x$.

21. $8A^6 - 27$.

12. $8r_1^3 - 27r_2^3$.

17. $125a^2 - a^5$.

22. $v^6 - 64w^3$.

13. $64g^3 - 27a^3$.

18. $2Pt^3 - 16P^4$.

23. $125 - Z^{12}$.

14. $216V^3 - 125V_1^3$.

19. $81y^4 - 24y$.

24. $27R^9 - V^3$.

15. $a^3b^3 - 1$.

20. $m^3 - n^6$.

25. $4\pi r^4 - 32\pi rh^3$.

26. The volume of a sphere whose diameter is D is $\frac{1}{6}\pi D^3$. Show that the volume of a spherical shell whose outer diameter

is D and inner diameter d , is $V = \frac{1}{6}\pi(D-d)(D^2 + Dd + d^2)$.



27. Compute, by the formula in Problem 26, the volumes of spherical shells whose respective outer and inner diameters are in the following table.

D	4 in.	6 in.	12 in.	15 in.
d	3 in.	4 in.	10 in.	12 in.

92. Binomials of the Form $a^3 + b^3$.—From § 82 it follows that the factors of $a^3 + b^3$ are $a + b$ and $a^2 - ab + b^2$. Test these factors by actual multiplication. Hence,

One factor of the sum of the cubes of two terms is the sum of the terms, and the other factor is the sum of their squares minus their product.

EXAMPLE.—Factor $27 W^3 + 64 V^3$.

$$\begin{aligned} 27 W^3 + 64 V^3 &= (3 W)^3 + (4 V)^3 \\ &= (3 W + 4 V)(9 W^2 - 12 WV + 16 V^2). \end{aligned}$$

Check.—When $W = 1$ and $V = 1$, the binomial is 91 and the factors 7 and 13, respectively.

EXERCISES

Factor:

- | | | |
|--------------------|------------------------------|---------------------------------|
| 1. $m^3 + n^3$. | 11. $64 N^3 + 1$. | 21. $27 P^2 + P^5$. |
| 2. $A^3 + 1$. | 12. $8 a^3 + b^3$. | 22. $8 x^4 + 125 x$. |
| 3. $1 + p^3$. | 13. $y^3 + 27 z^3$. | 23. $81 D^4 + 24 D$. |
| 4. $w^3 + 8$. | 14. $W^3 + 64 D^3$. | 24. $v^3 + t^6$. |
| 5. $H^3 + 64$. | 15. $27 S^3 + 8 T^3$. | 25. $8 N^6 + 1$. |
| 6. $27 + v^3$. | 16. $125 r_1^3 + 27 r_2^3$. | 26. $40 a^6 + 135 b^3$. |
| 7. $125 + t^3$. | 17. $64 a^3 + 27 t^3$. | 27. $\pi D^3 + \pi d^3$. |
| 8. $D^3 + 216$. | 18. $16 x^3 + 54 y^3$. | 28. $54 R_1^3 + 2 R_2^3$. |
| 9. $1 + 8 a^3$. | 19. $a^3 b^3 + 1$. | 29. $3\pi r^4 + 81 \pi r h^3$. |
| 10. $1 + 27 x^3$. | 20. $27 + m^3 n^3$. | 30. $w^6 + v^6$. |

93. General Suggestions on Factoring.—As shown in the preceding sections, the method of factoring any given expression depends upon the special form of the expression. There are no general rules or principles that can be laid down for factoring all expressions. To factor any expression, wherever the need for it is encountered, requires first that *the expression be identified as having some special form*, such as one of those discussed in this chapter, and second *the application of the special rule for factoring expressions of that form*. The following miscellaneous exercises

are given for practice in identifying the special forms of expressions to be factored.

General Suggestions:

I. *In factoring any expression, first see if there is a monomial factor; and if there is one, separate the expression into this monomial and the corresponding polynomial factor.*

II. *Then try to identify the polynomial factor as having one of the special forms discussed in the preceding sections; and if successful, factor it by the appropriate rule.*

III. *Repeat the process with each factor obtained, until the prime factors of the given expression are found. Write the given expression as the indicated product of all of the prime factors.*

MISCELLANEOUS EXERCISES

Factor:

- | | |
|--|---------------------------------|
| 1. $A^5 + 3 A^3 + A^2$. | 16. $M^2 + 13 M + 42$. |
| 2. $2 x^3 y + 6 x^2 y^2 + 4 x y^3$. | 17. $42 - 13 y + y^2$. |
| 3. $4 w^2 + 12 w v + 9 v^2$. | 18. $W^2 - 42 - W$. |
| 4. $m^2 n + 6 m n + 9 n$. | 19. $a^2 - 42 + a$. |
| 5. $P^2 + 8 P + 12$. | 20. $t^3 - 4 t^2 + 4 t$. |
| 6. $t^2 - 8 t + 12$. | 21. $8 - B^3$. |
| 7. $R^2 + 4 R - 12$. | 22. $x^3 + 2 x^2 - 48 x$. |
| 8. $n^2 - 4 n - 12$. | 23. $x^3 - 2 x^2 - 48 x$. |
| 9. $a^3 + 8 a^2 + 15 a$. | 24. $W^2 - 9 W g + 20 g^2$. |
| 10. $M^3 - 8 M^2 + 15 M$. | 25. $W^2 + 9 W g + 20 g^2$. |
| 11. $d^3 + 2 d^2 - 15 d$. | 26. $H^4 - 16 t^4$. |
| 12. $v^2 - v^4$. | 27. $a^3 + a^2 + 8 a + 8$. |
| 13. $5 x^3 y - 10 x^2 y^2 + 5 x y^3$. | 28. $1 - 4 m^2 + 4 m v - v^2$. |
| 14. $a - a^4$. | 29. $x^2 y + 3 x y - 28 y$. |
| 15. $D^3 - 2 D^2 - 15 D$. | 30. $72 + N^2 + 17 N$. |

- | | |
|---------------------------------|--|
| 31. $17 at + 72 t + a^2 t.$ | 59. $V^3 + V^2 - 42 V.$ |
| 32. $A^2 + 5 A - 50.$ | 60. $x^2 + 2 xy - 35 y^2.$ |
| 33. $A^2 - 5 A - 50.$ | 61. $a^4 - (a - 6)^2.$ |
| 34. $2 x^2 + 3 x + 1.$ | 62. $p^2 q^2 - p^2 - q^2 + 1.$ |
| 35. $t^3 - 125.$ | 63. $7 w^2 + 36 w + 5.$ |
| 36. $a^2 z^2 - b^2 w^2.$ | 64. $2 A^2 + 5 A - 3.$ |
| 37. $144 L^2 - M^4.$ | 65. $2 A^2 - 5 A + 3.$ |
| 38. $ax - bx - a + b.$ | 66. $a^2 + 4 ab - c^2 + 4 b^2.$ |
| 39. $v^2 - 15 v + 50.$ | 67. $t^3 + 125.$ |
| 40. $x^2 y^2 - 10 xy - 56.$ | 68. $5 m^2 - 9 mn - 2 n^2.$ |
| 41. $2 a^2 - 3 a + 1.$ | 69. $81 y^3 - 3.$ |
| 42. $Q^4 t^6 - 9 V^2.$ | 70. $12 H^2 + 5 H - 2.$ |
| 43. $xy - x - y + 1.$ | 71. $216 + v^3.$ |
| 44. $m^3 + n^6.$ | 72. $3 abx^2 + 2 abx - ab.$ |
| 45. $2 A^2 + 5 AB + 2 B^2.$ | 73. $8 D^3 + 1.$ |
| 46. $v^3 + 8.$ | 74. $6 bc - 11 st + 5 t^2.$ |
| 47. $w^2 v^2 + 5 wvt + 6 t^2.$ | 75. $81 x^4 - 100 v^4.$ |
| 48. $D^4 - 7 D^2 + 10.$ | 76. $N^2 + aN - 2 N - 2 a.$ |
| 49. $x^4 y^4 + 7 x^2 y^2 + 10.$ | 77. $4 x^2 - (a + b)^2.$ |
| 50. $V^4 + 11 V^2 - 26.$ | 78. $27 E^3 + 1.$ |
| 51. $6 a^2 + 11 a + 3.$ | 79. $4 A^2 + 32 AB + 39 B^2.$ |
| 52. $12 w^2 - 23 w + 10.$ | 80. $125 v^3 + t^3.$ |
| 53. $27 + T^3.$ | 81. $4(a - b)^2 - (y + z)^2.$ |
| 54. $36 M^2 - 64 N^2 M^2.$ | 82. $8 B^2 + 15 B - 2.$ |
| 55. $3 x^2 + 7 xy + 2 y^2.$ | 83. $r_1^2 + 6 r_1 r_2 - 1 - 4 r_3 - 4 r_3^2 + 9 r_2^2.$ |
| 56. $81 B^4 - 16 C^4.$ | 84. $40 M^3 + 135 V^3.$ |
| 57. $ab - 3 a - 2 b + 6.$ | 85. $H^2 - t^2 + vH - vt.$ |
| 58. $2 m^2 - 9 mn + 10 n^2.$ | 86. $3 x^2 + x - 2.$ |

87. $A^3 + A^2n - A - n.$

88. $15 + 2z - z^2.$

89. $xy + z^2 - xz - yz.$

90. $9x^2 - 27x + 20.$

91. $D^3 - 216.$

92. $W^6 - V^6.$

93. $mn^2 + 7mn - 30m.$

94. $1 + 4a^2t^2 - a^4 - 4t^4.$

95. $9a^2 + 6bd - d^2 - 9b^2.$

96. $v^3 - 216t^6.$

97. $5Q^2 - 79Qt - 16t^2.$

98. $x + x^4 - ax^3 - a.$

99. $v^3 + v^2w + vw^2 + w^3.$

100. $R^2 + 2R - Rr - 2r.$

HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

94. Highest Common Factor. — An expression that is a factor of each of two or more expressions is called a **common factor** of those expressions. The expression with the greatest numerical coefficient and the highest powers of its literal factors that is a common factor of two or more expressions is called their **highest common factor** (H.C.F.).

We shall consider here only the highest common factors of expressions whose factors may be obtained by the methods of this chapter.

EXAMPLE 1. — Find the H.C.F. of $8A^3BC^4$ and $12A^4B^2C^2$.

The largest number contained in both 8 and 12 is 4.

The highest power of A contained in both A^3 and A^4 is A^3 .

The highest power of B contained in both B and B^2 is B .

The highest power of C contained in both C^4 and C^2 is C^2 .

Hence, the H.C.F. is $4A^3BC^2$.

EXAMPLE 2. — Find the H.C.F. of $x^3 - x^2y - xy^2 + y^3$ and $2x^3 - 4x^2y + 2xy^2$.

$$x^3 - x^2y - xy^2 + y^3 = (x - y)^2 (x + y).$$

$$2x^3 - 4x^2y + 2xy^2 = 2x(x - y)^2.$$

Hence, H.C.F. = $(x - y)^2$, or $x^2 - 2xy + y^2$.

To find the highest common factor of two or more monomials, take the product of the greatest common divisor of their numerical coefficients and each letter raised to the lowest power to which it appears in any of the monomials.

In polynomials, factor each expression and proceed as with monomials, treating each factor as you would a single letter.

EXERCISES

Find the highest common factor of:

1. $4a^3b^2, 6a^2b^4.$
2. $9a^4b^3c^2, 12a^3b^4c^5.$
3. $15W^7V^6, 20W^4V^3U^2.$
4. $8mn^4p, 96m^3np^2.$
5. $16M^5N^5, 24M^2N^6P^2.$
6. $2x^{10}y^9z^2, y^6z^6.$
7. $49wv^4t^2s, 21w^4v^3t^3u.$
8. $35C^7DP^5, -56CD^2P^3.$
9. $24m^2np^4, 42mp^3, 18m^3p^2q.$
10. $12(v+t)^2, 8(v+t)^3.$
11. $14(E-F)^6, 21(E-F)^4.$
12. $6(x-1)^2, 8(x-1)(x-2).$
13. $10(y+1)^3, 4(y+1)^2(y-1).$
14. $39(m-n)^2(m+n)^3, 26(m-n)(m+n)^4.$
15. $A^2-1, A^2+2A-3.$
16. $1-t^2, 1-t^3, 1-t^4.$
17. $V^2+2V-15, V^2-4V+3.$
18. $a^2-b^2, a^3-b^3, a^2-3ab+2b^2.$
19. $2x^2+4x, 4x^3+12x^2+8x.$
20. $R^2-3R, 9-R^2, R^2-5R+6.$
21. $w^2-1, w^3+1, w+1.$
22. $2a^2+a-6, 6a^2-7a-3.$
23. $3r_1r_2^3-3r_1^3r_2, 4r_1^2r_2^3+2r_1^3r_2^2-6r_1^4r_2.$
24. $A^5-AB^4, A^3+A^2B+AB+B^2.$
25. $6m^3n+14m^2n^2+4mn^3, 4m^4+16m^3n+16m^2n^2.$

95. Lowest Common Multiple.—A multiple of a given expression is any expression of which the given expression is a factor. A **common multiple** of two or more expressions is an expression that is a multiple of each. The **lowest common multiple** (L. C. M.) of two or more given expressions is the expression with the least numerical coefficient and the lowest powers of its literal factors that is a common multiple of the given expressions.

EXAMPLE 1. — Find the L. C. M. of $24 A^2 B^3 C^4$ and $30 A^4 B^2 C^2$.

The least number that will contain both 24 and 30 is 120.

The lowest power of A of which A^2 and A^4 are factors is A^4 .

The lowest power of B of which B^3 and B^2 are factors is B^3 .

The lowest power of C of which C^4 and C^2 are factors is C^4 .

Hence, the L. C. M. is $120 A^4 B^3 C^4$.

EXAMPLE 2. — Find the L. C. M. of $2v^2 - 4v + 2$ and $3v^2 + 3v - 6$.

$$2v^2 - 4v + 2 = 2(v - 1)^2.$$

$$3v^2 + 3v - 6 = 3(v + 2)(v - 1).$$

$$\text{Hence, L. C. M.} = 6(v - 1)^2(v + 2).$$

To find the lowest common multiple of two or more monomials, take the product of the least common multiple of the numerical coefficients, and each letter raised to the highest power to which it appears in any one of the monomials.

In polynomials, factor each expression, and proceed as with monomials, treating each factor as you would a single letter.

EXERCISES

Find the lowest common multiple of :

1. $2a^2b$, $3ab^3$.
2. $6x^3y^2$, $10xy^4$.
3. $8w^6vu^3$, $12wv^2$.
4. $81M^2N^5$, $18MN^6$.
5. $7pq^3r$, $3p^2q$, $6p^3r^2$.
6. $21A^5BC^3$, $4A^3BC^2$, $6AB^3$.
7. $25m^2n^3$, $4amn^2$, $10a^2mn^2$.
8. $15R^5r^5$, $6R^2r^3$, $4R^3r^2$.
9. $(x + y)^3$, $2(x + y)^2$.
10. $(A - B)^3$, $3(A - B)(A + B)^2$.
11. $n(n - 1)(n + 1)$, $2n^2(n + 1)$.
12. $t^3(1 + t)^2(1 - t)$, $(1 + t)(1 - t)$.
13. $S^2 - 1$, $S^2 - 2S + 1$.
14. $D^2 - 1$, $D^2 - D - 2$.
15. $1 - y^2$, $y + y^2 - 2y^3$.
16. $P^2 - Q^2$, $2P^2 + 3PQ + Q^2$.
17. $t^2 + 5t + 6$, $t^2 + 7t + 12$.
18. $v^2 + v - 30$, $v^2 + 5v - 6$.
19. $2A^2 + 3A - 2$, $3A^2 + 7A + 2$.
20. $2x^2 - x - 10$, $2x^2 + x - 3$.
21. $2N^2 + 2N$, $N^2 + 5N + 6$.
22. $k^3 - 1$, $k^3 - k$, $2k^2$.

23. $P^4 - 1, P^2 + 3P + 2, P^2 + P - 2.$
24. $1 - d^2, 1 - d^3, 1 - d.$
25. $z^2 - z^4, 2z^2 - 4z, z^2 + 1.$
26. $B^2 + 5B - 14, 4B^3 - 16B^2 + 16B, 3B^2.$
27. $a^2 - b^2, a^3 - b^3, a^4 - b^4.$
28. $a^2 + ab + ac + bc, a^2 + 2ab + b^2.$
29. $x^2 - y^2, x^2 - xy + xv - yv, x^2 + xy + yv + xv.$
30. $A^4 - B^4, A^4 + 2A^2B^2 + B^4, A^4 - 2A^2B^2 + B^4.$

EQUATIONS SOLVED BY FACTORING

96. Equations solved by Factoring.—One important use of factoring is in the solution of equations of the second or higher degrees. In solving an equation by factoring, use is made of the principle that :

If one of the factors of an expression is zero, the whole expression must be zero.

EXAMPLE.—Solve $n^2 + n = 6$.

Transposing, $n^2 + n - 6 = 0$.

Factoring, $(n - 2)(n + 3) = 0$.

Now this equation is satisfied by any value of n that will make either of the factors $n - 2$ or $n + 3$ zero, since then their product must be zero. Such values of n are obtained by setting each factor equal to zero, and solving the resulting linear equations.

Setting each factor equal to zero,

$$n - 2 = 0; \text{ whence } n = 2.$$

$$n + 3 = 0; \text{ whence } n = -3.$$

Check.—When $n = 2$, the equation becomes $4 + 2 = 6$.

When $n = -3$, the equation becomes $9 - 3 = 6$.

To solve an equation by factoring:

(1) *Transpose all terms to the first member, making the second member 0.*

(2) *Factor the resulting first member.*

(3) *Set each of the factors equal to 0, and solve the resulting equations.*

It will be found that an equation of the second degree has two roots, an equation of the third degree three roots, etc.

NOTE.—Care must be exercised not to divide both members of an equation by an expression containing the unknown number, because in so doing roots are lost.

EXERCISES

Solve:

1. $n^2 - 6n + 8 = 0.$

2. $b^2 + 4b = 12.$

3. $x^2 - 9 = 0.$

4. $P^2 = 16.$

5. $4a^2 - 25 = 0.$

6. $9t^2 - 49 = 0.$

7. $S^2 = 64.$

8. $4 - r^2 = 0.$

9. $5B^2 + 6 = 4B^2 + 7.$

10. $7d^2 - 28 = 0.$

11. $A(A + 4) = 4A + 49.$

12. $c^2 = 8c + 9.$

13. $t^2 = 22 + 9t.$

14. $V^2 - 15V = 54.$

15. $D^2 + 12D + 20 = 0.$

16. $p^2 = 6 - p.$

17. $W^2 + 14W + 40 = 0.$

18. $h^2 - 11h + 30 = 0.$

19. $N^2 + 5N = 14.$

20. $y^2 + 12 = 7y.$

21. $10 = 3A + A^2.$

22. $6k = 9 + k^2.$

23. $Q^2 = 3Q + 54.$

24. $2x^2 + 2 = 5x.$

25. $2p^2 + 5p = 3.$

26. $5B^2 = 1 + 4B.$

27. $21w^2 = 10 + 29w.$

28. $x^3 - x = 0.$

29. $m^3 + 2m^2 = 3m.$

30. $V^3 = 3V^2 + 10V.$

31. $4t^3 + 10t^2 + 4t = 0.$

32. $a^3 + a^2 = 9a + 9.$

33. $R^3 + 4 = R^2 + 4R.$

34. $3p^3 - p^2 = 27p - 9.$

35. $32 + x^3 = 16x + 2x^2.$

36. $N^4 - 5N^2 + 4 = 0.$

37. $v^4 = 9v^2.$

38. $A^4 + 12A^2 = 7A^3.$

39. $t^4 - 18t^2 + 81 = 0.$

40. $2D^4 + 18 = 20D^2.$

41. Explain the fallacy in the following reasoning, by which it is shown that

$$2 = 1.$$

Let	$n = 1.$
Multiplying by n ,	$n^2 = n.$
Hence,	$n^2 - 1 = n - 1.$
Dividing by $n - 1$,	$n + 1 = 1.$
Substituting 1 for n ,	$2 = 1.$

42. If to the square of a certain number five times the number is added, the sum is 104. Find the number.

43. Find two consecutive whole numbers whose product is 182.

44. The sum of the squares of two consecutive odd numbers is 202. Find the numbers.

45. The sum of the squares of three consecutive numbers is 194. Find the numbers.

46. The area of a triangle is 144 sq. in., and the altitude is twice as long as the base. Find the base and altitude.

47. The base of a triangle is 7 ft. longer than the altitude, and the area is 60 sq. ft. Find the base and altitude.

48. The base of a right triangle is 3 ft. greater than the altitude, and the hypotenuse is 15 ft. Find the length of the base and altitude.

49. One base of a trapezoid is 2 in. greater than the altitude, and the other base is 4 in. greater than the altitude. The area is 54 sq. in. Find the bases and the altitude.

50. The altitude of a trapezoid equals one base and exceeds the other base by 4 in. The area is 120 sq. in. Find the bases and altitude.



51. A square box without a lid and 6 in. deep is to be made from a square piece of tin by cutting out a square from each corner, bending up the sides and soldering along the edges. The box is to hold 384 cu. in.

Find how large to cut the piece of tin from which to make the box.

52. An open box, to be 4 in. deep, twice as long as wide, and to hold 512 cu. in., is to be made from a rectangular piece of cardboard by cutting out a square from each corner, bending up the sides and pasting along the edges. Find how large to cut the cardboard from which to make it.

53. A farmer has a field 60 rd. wide and 80 rd. long, which he is plowing to sow with wheat. How wide a strip must he plow around the field in order to have it half done?

54. Two boys have a lawn to mow that is 60 ft. long and 32 ft. wide. The first boy is to mow half of it by cutting a strip of uniform width around it, and the other is to finish it. How wide a strip must the first boy cut?

55. By measuring the side of a room 15 ft. square, a man found that he got an area 3500 sq. in. too small. Find the amount of his error in measuring the side of the room.

SUGGESTION.—If e inches is the error, show that $e^2 - 360e + 3500 = 0$.

56. A room is 10 ft. square. What error (in inches) in measuring the side would make the computed area 476 sq. in. too small?

57. In the room in Problem 56, what error in measuring the side would make the computed area 729 sq. in. too large?

58. The horse power of a gasoline engine is computed by the formula $H = \frac{D^2 N}{2.5}$, where H = horse power, D = diameter of cylinders in inches, and N = number of cylinders. In an engine of one cylinder, find the diameter of the cylinder required to yield 40 horse power.

59. In a gasoline engine of 4 cylinders, find the diameter of the cylinders required to yield 160 horse power.

60. From a group of n children two leaders for a game may be selected in $n(n-1)$ ways. If the number of ways in which the two leaders may be chosen is 240, how many children are there in the group?

SUPPLEMENTARY EXERCISES

Factor :

- ✓ 1. $4 A^2 B^3 - 8 B^5$.
 ✓ 2. $ax^n + bx^n$.
 ✓ 3. $v^n - wv^n$.
 4. $a^m t + a^{m-1} t^2$.
 ✓ 5. $P^{2x} - 5 P^x Q^x$.
 6. $r^{n-1} s^{n+1} + r^{n+1} s^{n-1}$.
 • 7. $5 v^a y^{a-1} - 10 v^{a-1} y^a$.
 ✓ 8. $8 M^{2n+1} + 12 M^{2n} N$.
 ✓ 9. $a^2 b^2 - (a - b)^2$.

10. $m^2 - n^2 + mx - nx$.

SUGGESTION. — Group and factor the first two terms, then group and factor the last two terms.

11. $V^2 - W^2 + tV - tW$.

12. $x - a + (x - a)^2$.

13. $x^2 - x - y^2 + y$.

14. $m^4 n^4 - m^4 - n^4 + 1$.

15. $r_1^4 - r_2^4 - (r_1^2 - r_2^2)^2$.

16. $At^2 + At + Bt^2 + Bt + A + B$.

SUGGESTION. — Group the first, second, and fifth terms; also group the third, fourth, and sixth terms. One factor is a trinomial.

17. $2a - 2b + ad - bd + 2c + cd$.

18. $ny^2 + 3ny + 5y^2 + 15y + 2n + 10$.

19. $p^3 + 4p^2 + 4p - p^2q - 4pq - 4q$.

20. $x^2 - y^2 + z^2 - w^2 - 2(xz - yw)$.

21. $W^4 + W^2 + V^2 - V^4$.

22. $A^3 + A^2C + ABC + B^2C - B^3$.

SUGGESTION. — Group the first and last terms together, and the other three terms together.

23. $4m^2n^2 - (m^2 + n^2 - p^2)^2$.

24. $a(a + c) - b(b + c)$.

25. $w^2 - v^2 + u(u - 2w)$.

26. $x^{2n} - 2x^n y^n + y^{2n}$.

27. $a^n - 6a^n t^n + 9a^n t^{2n}$.

28. $6P - 9P^2 - 1$.

SUGGESTION. — First remove the monomial factor -1 .

29. $-t^4 + 8t^2 - 16$.

30. $-x^2 + 2xy^2 - y^4$.

31. $-4M^2 - 12MN - 9N^2$.

32. $(a + b)^2 + 7(a + b) + 10$.

33. $(x - y)^2 - 6(x - y) - 40$.

34. $ax^2 + (a + b)x + b$.

35. $ax^2 + (b - a)x - b$.

36. $2y^2 + (4a + b)y + 2ab.$

37. $2W^2 - (2m + n)W + mn.$

38. $av^2 + (ab + 1)v + b.$

39. $x^{2n} - y^{2n}.$

40. $4A^{4n} - 9B^{6n}.$

41. $w^{3n} + v^{3n}.$

42. $1 - Q^{3n}.$

43. $x^6 - y^6.$

SUGGESTION.—First factor as the difference of two squares, then factor the factors.

$$x^6 - y^6 = (x^3 - y^3)(x^3 + y^3) = \text{etc.}$$

44. $1 - n^6.$

45. $D^6 - 1.$

46. $a^6 - 64b^6.$

47. $R^{12} - r^{12}.$

Solve the following equations for x :

48. $x^2 - 3ax + 2a^2 = 0.$

52. $x^2 - w^2 = 2vw + v^2.$

49. $n^2x^2 - n^4 = 0.$

53. $x^2 + 9 + 4ab = 6x + a^2 + 4b^2.$

50. $4vx = x^2 + 4v^2.$

54. $x^2 = 2rx + 63r^2.$

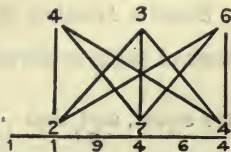
51. $2x^2 - 3t^2 = 5xt.$

55. $x^3 + nx^2 = b^2x + b^2n.$

56. Any two arithmetical numbers with three figures each may be written in the polynomial forms $100C + 10B + A$ and $100c + 10b + a$, respectively. Show that their product is $10000Cc + 1000(Cb + cB) + 100(Ca + Bb + Ac) + 10(Ab + aB) + Aa$.

From the form of this expression may be deduced a process, called "cross-multiplication," for finding the product of two arithmetical numbers in an abbreviated form. The process is applicable to numbers with any number of figures. By this process practically all of the work in multiplication is done without the aid of pencil, but it requires great mental concentration.

Thus, to find 274×436 the work is done mentally as follows, and only the answer written: To get the ones, take $4 \times 6 = 24$ and write the 4; to get the tens, take $2 + 4 \times 3 + 6 \times 7 = 56$ and write the 6; to get the hundreds, take $5 + 4 \times 4 + 6 \times 2 + 7 \times 3 = 54$ and write the 4; then take $5 + 7 \times 4 + 3 \times 2 = 39$, and write the 9; then take $3 + 2 \times 4 = 11$.



57. Find, by "cross-multiplication," the products of: 34×52 ; 61×36 ; 47×54 ; 67×62 ; 52×83 ; 245×326 ; 318×241 ; 234×571 ; 384×615 ; 724×318 ; 27×356 (same as 027×356).

CHAPTER IX

FRACTIONS

97. Algebraic Fractions. — The indicated quotient of two numbers or algebraic expressions, as $\frac{m}{n}$ or $\frac{a^2 - b^2}{a^2 + 2ab + b^2}$, is called a **fraction**. In algebra, as in arithmetic, the dividend is called the **numerator**, and the divisor the **denominator**, of the fraction. The numerator and denominator are called the **terms** of the fraction.

The fraction $\frac{2x - 1}{x^2 + x + 1}$ is read either “ $2x - 1$ divided by $x^2 + x + 1$ ” or, more briefly, “ $2x - 1$ over $x^2 + x + 1$.”

98. Signs of a Fraction. — In an algebraic fraction there are three signs involved: (1) the *sign before the fraction*, which is the plus or minus sign written (or understood) before the line separating the numerator from the denominator; (2) the *sign of the numerator*; (3) the *sign of the denominator*.

Since a fraction is an indicated quotient, it follows from the laws of signs in division that $\frac{+a}{+b}$ and $\frac{-a}{-b}$ are both positive and are equal, and that $\frac{-a}{+b}$ and $\frac{+a}{-b}$ are both negative and are equal.

Hence,

$$\frac{+a}{+b} = \frac{-a}{-b} = -\frac{-a}{+b} = -\frac{+a}{-b}.$$

It follows that to preserve the algebraic value of a fraction

(1) *If the signs of both terms of the fraction are changed, the sign before the fraction must be left unchanged.*

(2) If the sign of only one term of the fraction is changed, the sign before the fraction must be changed.

$$\text{Thus, } \frac{+12}{+4} = \frac{-12}{-4} = -\frac{-12}{+4} = -\frac{+12}{-4} = 3.$$

If the numerator or denominator be a polynomial, its sign is changed by changing the sign of each of its terms, because this is equivalent to multiplying the polynomial by -1 .

$$\text{Thus, } \frac{x^2 - 4x + 2}{-x^2 + 3x - 1} = \frac{-x^2 + 4x - 2}{x^2 - 3x + 1} \text{ or } -\frac{x^2 - 4x + 2}{x^2 - 3x + 1}.$$

EXERCISES

Write with positive numerators and denominators:

1. $\frac{-3}{8}.$

4. $\frac{10}{-12}.$

7. $\frac{-2a^2}{10b^2}.$

10. $\frac{d}{-M^3}.$

2. $\frac{-2}{5}.$

5. $\frac{-6}{-7}.$

8. $\frac{7n}{-9t^2}.$

11. $\frac{-100}{P^4}.$

3. $\frac{4}{-9}.$

6. $\frac{-3x}{-5y}.$

9. $\frac{-1}{-V^2}.$

12. $\frac{-at^2}{-vg^2}.$

Write with denominators $a - b$:

13. $\frac{1}{b - a}.$

14. $\frac{-x}{b - a}.$

15. $\frac{m - n}{b - a}.$

In each of the following expressions write all of the fractions with the same denominator:

16. $\frac{6}{x - y} + \frac{2}{y - x}.$

18. $\frac{2x + 1}{x^2 - 1} - \frac{3x}{1 - x^2} + \frac{1 - x}{1 - x^2}.$

17. $\frac{a}{a - b} + \frac{b}{b - a} - \frac{a + b}{b - a}.$

19. $\frac{5at}{a^2 - t^2} - \frac{a^2}{t^2 - a^2} + \frac{a^2 + t^2}{a^2 - t^2}.$

20. $\frac{1 + r}{1 - r} - \frac{r}{r - 1} + \frac{1}{r - 1}.$

Write for each of the following an equal fraction with positive denominator and the sign + before the fraction :

$$\begin{array}{llll} 21. & -\frac{x}{-y} & 23. & -\frac{2x}{-3y} & 25. & -\frac{-1}{-4} & 27. & \frac{\pi r^2}{-\pi r^3} \\ 22. & \frac{w}{-v} & 24. & -\frac{-R}{-N} & 26. & \frac{-5}{-n} & 28. & \frac{-vg^2}{-100} \end{array}$$

99. Reduction of Fractions to Lowest Terms. — As in arithmetic, to change a fraction to lowest terms is to change it to a fraction with equal value, but whose numerator and denominator have no common factor.

Thus, $\frac{6}{8} = \frac{2 \times 3}{2 \times 4} = \frac{3}{4}$, $\frac{9}{18} = ?$, $\frac{16}{24} = ?$, $\frac{36}{100} = ?$

The principle involved in the process is that :

Dividing both terms of a fraction by the same expression does not change the value of the fraction.

EXAMPLE 1. — Reduce $\frac{12 a^2 b^6}{18 a^3 b^4}$ to lowest terms.

Dividing both terms by $6 a^2 b^4$, their highest common factor,

$$\frac{12 a^2 b^6}{18 a^3 b^4} = \frac{2 b^2}{3 a}$$

EXAMPLE 2. — Reduce $\frac{x^2 + 2x - 3}{x^2 + 5x + 6}$ to lowest terms.

Factoring both terms, and dividing by $x + 3$, their H. C. F.,

$$\frac{x^2 + 2x - 3}{x^2 + 5x + 6} = \frac{(x + 3)(x - 1)}{(x + 3)(x + 2)} = \frac{x - 1}{x + 2}$$

It is evident that :

To reduce an algebraic fraction to its lowest terms, factor the numerator and denominator, and divide both terms by their highest common factor.

The division may be indicated by *cancellation* of the common factors of the terms.

$$\text{Thus, } \frac{am^2 - an^2}{am^3 - an^3} = \frac{\cancel{a(m-n)}(m+n)}{\cancel{a(m-n)}(m^2 + mn + n^2)} = \frac{m+n}{m^2 + mn + n^2}$$

NOTE.—It must be remembered in using cancellation that only *factors* common to the numerator and denominator may be cancelled. *Terms* in polynomial numerators and denominators cannot be cancelled—a common mistake made by students. Thus, in $\frac{3+5}{4+5}$ the 5's cannot be cancelled, because the fraction equals $\frac{8}{9}$, and if the 5's were cancelled, it would become $\frac{3}{4}$. Similarly, in $\frac{x^2+1}{x^2+4}$ no cancellation is possible.

EXERCISES

Reduce to lowest terms:

- | | | | |
|---|--|--------------------------------------|--------------------------------|
| 1. $\frac{10}{12}$ | 6. $\frac{36}{96}$ | 11. $\frac{ab}{ac}$ | 16. $\frac{5xyz^2}{10yz}$ |
| 2. $\frac{9}{12}$ | 7. $\frac{42}{54}$ | 12. $\frac{3x}{6x^2}$ | 17. $\frac{3w^2v^3}{12wv^4}$ |
| 3. $\frac{15}{18}$ | 8. $\frac{72}{100}$ | 13. $\frac{4m^2n}{6mn^2}$ | 18. $\frac{7rst^2}{21r^2st}$ |
| 4. $\frac{21}{28}$ | 9. $\frac{240}{800}$ | 14. $\frac{12p^4q^2}{16p^3q^5}$ | 19. $\frac{8Mv^2}{36v^5}$ |
| 5. $\frac{12}{20}$ | 10. $\frac{60}{720}$ | 15. $\frac{4abc}{6a^2bd}$ | 20. $\frac{4\pi R^2}{\pi R^3}$ |
| 21. $\frac{4\pi R(R+H)}{8\pi R^2(R+H)}$ | 25. $\frac{A^3 - AB^2}{A^2 + 2AB + B^2}$ | 29. $\frac{4n^2 + 4n + 1}{4n^2 - 1}$ | |
| 22. $\frac{a^2 - b^2}{a^2 + 2ab + b^2}$ | 26. $\frac{R^2 - 1}{2Ry + 2y}$ | 30. $\frac{a^2 - b^2}{a^3 - b^3}$ | |
| 23. $\frac{xn - xm}{n^2 - m^2}$ | 27. $\frac{t^2 + 2t - 15}{t^2 - t - 6}$ | 31. $\frac{1-x}{1-x^3}$ | |
| 24. $\frac{v^2t - t^3}{v^4 - t^4}$ | 28. $\frac{1-p^2}{1-2p+p^2}$ | 32. $\frac{v^2 - w^2}{(v+w)^2}$ | |

33. $\frac{V^2 + 8V + 12}{V^3 + 6V^2}$. 35. $\frac{M^5 - M^3N^2}{M^4 - N^4}$. 37. $\frac{r_1^3r_2 - r_1r_2^3}{r_1^3r_2^2 + r_1^2r_2^3}$.
34. $\frac{ab + b^2}{ac^2 + bc^2}$. 36. $\frac{r^2 - 1}{r^2 + 3r + 2}$. 38. $\frac{m^4 - n^4}{m^4 + 2m^2n^2 + n^4}$.
39. $\frac{D^2 - D - 2}{4 - D^2}$. 42. $\frac{6x^2 + 7x - 5}{3x^2 + 17x + 20}$.
40. $\frac{ac - bc - ad + bd}{ac + ad - bc - bd}$. 43. $\frac{10a^2 - 33at - 7t^2}{12a^2 - 52at + 35t^2}$.
41. $\frac{R^2 - Rr - R + r}{R^2 - Rr - 2R + 2r}$. 44. $\frac{1 + P^3}{P^2 + 8P + 7}$.

100. Reduction of Fractions to Mixed Expressions. — An algebraic expression containing no fractions with literal denominators is an **integral expression**; an expression consisting of one or more such fractions is a **fractional expression**; and an expression consisting of an integral part and a fractional part is a **mixed expression**.

Thus, $x^2 - \frac{x-1}{x^2+1}$ and $\frac{A}{A-1} - \frac{A}{A+1} + A^2 - 1$ are mixed expressions.

NOTE. — In arithmetic the mixed number $4\frac{3}{8}$ means $4 + \frac{3}{8}$, the sign $+$ being omitted. In algebra the sign between the integral and fractional parts of a mixed expression must never be omitted. Thus, $x + \frac{y}{z}$ may not be written $x\frac{y}{z}$, because the latter implies multiplication.

An algebraic fraction whose numerator is of as high degree as the denominator, or of higher degree, may often be reduced to a mixed expression. The process is similar to that in arithmetic of reducing an improper fraction to a mixed number.

EXAMPLE 1. — Reduce $\frac{731}{46}$ to a mixed number.

Dividing 731 by 46 gives a quotient 15 and remainder 41.

Hence, $\frac{731}{46} = 15\frac{41}{46}$.

EXAMPLE 2. — Reduce $\frac{x^2 + 5x - 9}{x + 3}$ to a mixed expression.

Dividing $x^2 + 5x - 9$ by $x + 3$ gives a quotient $x + 2$ and remainder -15 .

Hence, $\frac{x^2 + 5x - 9}{x + 3} = x + 2 + \frac{-15}{x + 3}$.

This answer may be written $x + 2 - \frac{15}{x + 3}$.

To reduce a fraction to a mixed expression, divide the numerator by the denominator. The mixed expression equals the quotient plus the fraction whose numerator is the remainder and denominator the denominator of the given fraction.

EXERCISES

Reduce to mixed numbers:

- | | | | | |
|---------------------|----------------------|-----------------------|------------------------|--------------------------|
| 1. $\frac{7}{3}$. | 3. $\frac{38}{9}$. | 5. $\frac{126}{84}$. | 7. $\frac{192}{81}$. | 9. $\frac{1264}{150}$. |
| 2. $\frac{24}{5}$. | 4. $\frac{98}{17}$. | 6. $\frac{347}{86}$. | 8. $\frac{640}{280}$. | 10. $\frac{2437}{652}$. |

Reduce to mixed expressions:

- | | | |
|--------------------------------|-----------------------------------|-------------------------------------|
| 11. $\frac{x^2+1}{x-1}$. | 16. $\frac{P^2}{P-3}$. | 21. $\frac{9A^2+3AB+4B^2}{3A-2B}$. |
| 12. $\frac{2A^2-3A-7}{2A-4}$. | 17. $\frac{M^2+N^2}{M+N}$. | 22. $\frac{v^3-t^3}{v+t}$. |
| 13. $\frac{n^2-5n+1}{n-2}$. | 18. $\frac{r^3+3r^2+5r+2}{r+1}$. | 23. $\frac{n^3-3n^2+3n-1}{n+1}$. |
| 14. $\frac{1-x^3}{1+x-x^2}$. | 19. $\frac{D^3+16}{D+2}$. | 24. $\frac{S^5-16}{S-2}$. |
| 15. $\frac{w^3+8}{w^2-w+2}$. | 20. $\frac{32+a^4}{a-2}$. | 25. $\frac{E^3+4E^2+3}{E^2-3E}$. |

101. Reduction of Fractions to Higher Terms.—It sometimes is necessary to change a fraction to an equivalent fraction with denominator of higher degree. The principle involved is that:

If both terms of a fraction are multiplied by the same expression, the value of the fraction is not changed.

The process is the same as that in arithmetic of reducing a fraction to higher terms.

EXAMPLE 1. — Reduce $\frac{7}{8}$ to 48ths.

Dividing 48 by 8 gives 6. Hence, multiplying both terms by 6,

$$\frac{7}{8} = \frac{42}{48}.$$

EXAMPLE 2. — Change $\frac{a+b}{2a+b}$ to a fraction with denominator $6a^2+ab-b^2$.

Dividing $6a^2+ab-b^2$ by $2a+b$ gives $3a-b$. Hence, multiplying both terms by $3a-b$ gives

$$\frac{a+b}{2a+b} = \frac{3a^2+2ab-b^2}{6a^2+ab-b^2}.$$

Divide the required denominator by the given denominator, and multiply both terms of the fraction by the quotient.

NOTE. — The division may be quickly performed by factoring the required denominator and cancelling from it the factors of the given denominator. Thus, in Example 2, the required denominator $6a^2+ab-b^2 = (2a+b)(3a-b)$. Cancelling $2a+b$, the quotient is $3a-b$.

EXERCISES

1. Change $\frac{3}{5}$ to 20ths. 3. Change $\frac{9}{16}$ to 64ths.

2. Change $\frac{6}{7}$ to 56ths. 4. Change $\frac{3}{32}$ to 64ths.

5. Change $\frac{5}{18}$ to 126ths.

6. Change each of the following to 72ds:

$$\frac{1}{2}, \quad \frac{3}{4}, \quad \frac{5}{6}, \quad \frac{7}{8}, \quad \frac{11}{12}, \quad \frac{17}{18}.$$

7. Change $\frac{a}{b^2c}$ to a fraction with denominator $a^3b^3c^3$.

8. Change $\frac{1}{x-1}$ to a fraction with denominator x^2-1 .

9. Change $\frac{M-N}{M+N}$ to a fraction with denominator M^2-N^2 .

10. Change $\frac{R}{R+1}$ to a fraction with denominator R^2+3R+2 .

11. Change $\frac{v+t}{v-t}$ to a fraction with denominator v^3-t^3 .

12. Change $\frac{x}{2x+5y}$ to a fraction with denominator $6x^2+7xy-20y^2$.

13. Change each of the following to a fraction with denominator $a^3b - ab^3$:

$$\frac{1}{a+b}, \quad \frac{1}{a-b}, \quad \frac{1}{a}, \quad \frac{1}{b}.$$

14. Change each of the following to a fraction with denominator $(x-y)^2(x+y)^2$:

$$\frac{x}{x-y}, \quad \frac{y}{x+y}, \quad \frac{x+y}{(x-y)^2}, \quad \frac{x-y}{(x+y)^2}.$$

102. Reduction of Fractions to a Common Denominator.—It is clear that by the process of § 101 two or more fractions may be changed to equivalent fractions having for the denominator of each the lowest common multiple of all of the denominators of the given fractions. This lowest common multiple of the denominators of the given fractions is called the **lowest common denominator (L. C. D.)** of the fractions.

EXAMPLE.—Reduce $\frac{x+3}{x^2-1}$ and $\frac{1-x}{x^2-4x-5}$ to equivalent fractions having the lowest common denominator.

$$\begin{aligned} x^2-1 &= (x+1)(x-1). \\ x^2-4x-5 &= (x+1)(x-5). \end{aligned}$$

Hence,

$$\text{L. C. D.} = (x+1)(x-1)(x-5).$$

Cancelling the factors of the first denominator that are found in the L. C. D., it is seen that the terms of the first fraction must be multiplied by $x-5$. Similarly, the terms of the second fraction must be multiplied by $x-1$.

$$\text{Hence,} \quad \frac{x+3}{x^2-1} = \frac{(x-5)(x+3)}{(x-5)(x^2-1)} \text{ or } \frac{x^2-2x-15}{x^3-5x^2-x+5};$$

$$\text{and} \quad \frac{1-x}{x^2-4x-5} = \frac{(x-1)(1-x)}{(x-1)(x^2-4x-5)} \text{ or } \frac{-x^2+2x-1}{x^3-5x^2-x+5}.$$

EXERCISES

Reduce to equivalent fractions having the least common denominator:

$$1. \frac{1}{2}, \frac{2}{3}, \frac{3}{4}. \quad 3. \frac{3}{4}, \frac{5}{8}, \frac{2}{5}. \quad 5. \frac{8}{9}, \frac{3}{7}, \frac{5}{21}. \quad 7. \frac{1}{2}, \frac{3}{5}, \frac{7}{16}, \frac{5}{12}.$$

$$2. \frac{1}{3}, \frac{5}{6}, \frac{7}{8}. \quad 4. \frac{7}{8}, \frac{5}{9}, \frac{11}{12}. \quad 6. \frac{2}{15}, \frac{3}{20}, \frac{3}{4}. \quad 8. \frac{3}{16}, \frac{9}{40}, \frac{4}{5}, \frac{2}{3}.$$

Reduce to equivalent fractions having the lowest common denominator:

$$9. \frac{2}{3x}, \frac{1}{2x^2}, \frac{5}{6x}$$

$$12. \frac{z}{xy}, \frac{x}{yz}, \frac{y}{xz}$$

$$10. \frac{3}{8t^3}, \frac{5}{6t^3}, \frac{1}{4t^4}$$

$$13. \frac{A}{A+1}, \frac{A}{A-1}, \frac{1}{A^2-1}$$

$$11. \frac{2a^2}{3b^2c}, \frac{3b^2}{2a^2c}, \frac{c^2}{4a^2b^2}$$

$$14. \frac{3}{n^2+3n+2}, \frac{5}{2n^2+5n+2}$$

$$15. \frac{V-1}{V^2+5V+4}, \frac{V+3}{V^2+V-12}$$

$$16. \frac{2R+7}{6R^2+13R-5}, \frac{3R-4}{12R^2-13R+3}$$

$$17. \frac{1}{a-b}, \frac{2}{a+b}, \frac{1}{a^2+b^2}$$

$$18. \frac{2}{c+1}, \frac{c}{(c+1)^2}, \frac{2c^2}{(c+1)^3}$$

$$19. \frac{w}{v}, \frac{w^2}{wv-v^2}, \frac{wv}{w^2-v^2}$$

$$20. \frac{x}{x-y}, \frac{y}{2x-2y}, \frac{x^2}{6(x^2-y^2)}$$

$$21. \frac{p-5}{p^2+3p+2}, \frac{p+1}{p^2-3p-10}, \frac{p+2}{p^2-4p-5}$$

$$22. \frac{1}{6S^2+S-2}, \frac{1}{15S^2+10S}, \frac{1}{12S^2-6S}$$

103. Addition and Subtraction of Fractions.—Algebraic fractions are added or subtracted by the same rules as fractions are added or subtracted in arithmetic.

EXAMPLE 1.—Add $\frac{2}{3}$, $\frac{3}{8}$, and $\frac{5}{6}$.

Reducing all of the fractions to the least common denominator,

$$\frac{2}{3} + \frac{3}{8} + \frac{5}{6} = \frac{16}{24} + \frac{9}{24} + \frac{20}{24}.$$

Adding the numerators and placing the sum over the common denominator,

$$\frac{16}{24} + \frac{9}{24} + \frac{20}{24} = \frac{45}{24} \text{ or } 1\frac{3}{8}.$$

EXAMPLE 2.—Add $\frac{2x+1}{x-1}$, $\frac{x-2}{x+1}$, and $\frac{3}{x}$.

Reducing the fractions to their lowest common denominator,

$$\frac{2x+1}{x-1} + \frac{x-2}{x+1} + \frac{3}{x} = \frac{2x^3+3x^2+x}{x^3-x} + \frac{x^3-3x^2+2x}{x^3-x} + \frac{3x^2-3}{x^3-x}.$$

Adding the numerators, this becomes $\frac{3x^3+3x^2+3x-3}{x^3-x}$.

EXAMPLE 3.—Perform the indicated addition and subtraction in

$$\frac{5}{a^2+5a+4} + \frac{1}{a^2-a-2} - \frac{3}{a^2+2a-8}.$$

$$\frac{5}{a^2+5a+4} + \frac{1}{a^2-a-2} - \frac{3}{a^2+2a-8}$$

$$= \frac{5}{(a+1)(a+4)} + \frac{1}{(a-2)(a+1)} - \frac{3}{(a+4)(a-2)}$$

$$= \frac{5a-10}{(a+1)(a+4)(a-2)} + \frac{a+4}{(a+1)(a+4)(a-2)} - \frac{3a+3}{(a+1)(a+4)(a-2)}$$

$$= \frac{3a-9}{(a+1)(a+4)(a-2)} \text{ or } \frac{3a-9}{a^3+3a^2-6a-8}.$$

After reducing the fractions to a common denominator, the first two numerators are added, and the third one subtracted from their sum.

To add or subtract algebraic fractions, reduce them to their lowest common denominator, then find the sum or difference of their numerators, and place the result over the common denominator.

EXERCISES

Simplify by adding or subtracting, as indicated:

1. $\frac{3}{4} + \frac{5}{6}$.
2. $\frac{7}{8} - \frac{5}{12}$.
3. $\frac{9}{20} + \frac{7}{10} - \frac{3}{4}$.
4. $\frac{5}{18} + \frac{1}{6} - \frac{1}{12}$.
5. $\frac{5}{7} - \frac{1}{3} + \frac{3}{4}$.
6. $\frac{4}{21} - \frac{1}{14} + \frac{9}{20}$.
7. $\frac{7}{9} - \frac{1}{15} - \frac{1}{6}$.
8. $\frac{11}{12} + \frac{8}{9} - \frac{15}{16}$.
9. $\frac{3a}{4} + \frac{5a}{6} + \frac{a}{3}$.
10. $\frac{P}{2} + \frac{P}{3} + \frac{P}{4}$.
11. $\frac{3v}{20} + \frac{v}{5}$.
12. $\frac{4R}{7} + \frac{R-2}{5}$.
13. $\frac{a}{b} - \frac{c}{d}$.
14. $\frac{1}{A} + \frac{A-1}{A^2} + \frac{3}{2A}$.
15. $\frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy}$.
16. $\frac{n+1}{n^2} + \frac{1}{7n} + \frac{2n-1}{2n^2}$.
17. $\frac{1}{r_1 r_2} + \frac{1}{r_1 r_3} + \frac{1}{r_2 r_3}$.
18. $\frac{3}{4\pi R} + \frac{2\pi R + H}{2\pi R^2} + \frac{H + R}{\pi R^2}$.
19. $\frac{x}{x+y} + \frac{y}{x-y}$.
20. $\frac{1}{1+a} - \frac{1}{1-a}$.
21. $\frac{m+n}{m-n} + \frac{m-n}{m+n}$.
22. $\frac{v-1}{v+1} - \frac{v+1}{v-1}$.
23. $\frac{3}{s^2-s} - \frac{2}{s^2+s} + \frac{4}{s}$.
24. $\frac{2x+1}{x-2} - \frac{x+2}{2x-1}$.
25. $\frac{3v}{w+v} - \frac{v}{w-v} + \frac{2wv}{w^2-v^2}$.
26. $\frac{1}{a^2+6a+5} - \frac{1}{a^2+2a-15}$.
27. $\frac{M-2N}{M^2+5MN+4N^2} + \frac{M+4N}{M^2-MN-2N^2}$.
28. $\frac{1}{2b^2+5bc-3c^2} - \frac{1}{4b^2-13bc+3c^2}$.

$$29. \frac{2\pi R}{\pi R + \pi H} + \frac{2\pi R}{\pi R - \pi H}.$$

$$30. \frac{3}{y^2 + 2yz + z^2} - \frac{2}{2y^2 + yz - 10z^2}.$$

$$31. \frac{a+2}{a+1} + \frac{a-2}{a-1} + \frac{4}{1-a^2}.$$

SUGGESTION. — In problems such as this it is best to arrange the terms of all denominators according to either the descending or ascending powers of some letter. In order to have the first term of the denominator of each fraction positive when rearranged, it may be necessary to change the signs of the terms of the fraction, as in § 98. Thus, the last term in this problem may be changed to $+\frac{-4}{a^2-1}$.

$$32. \frac{P}{1-P^2} - \frac{1}{P-1}.$$

$$34. \frac{5}{1+x} + \frac{x}{1-x^2} + \frac{2x}{x^2-1}.$$

$$33. \frac{3v}{u+v} - \frac{v}{u-v} + \frac{2uv}{v^2-u^2}.$$

$$35. \frac{M}{M^2-N^2} - \frac{1}{M+N} - \frac{1}{N-M}.$$

$$36. \frac{3+2y}{2-y} + \frac{16y-y^2}{y^2-4} - \frac{2-3y}{2+y}.$$

$$37. \frac{R}{1+R} - \frac{R}{1-R} + \frac{R^2}{R^2-1}.$$

$$38. \frac{b^3}{1-b^3} - \frac{2b}{b^2+b+1} + \frac{1}{b-1}.$$

$$39. \frac{x+2a}{2b-x} + \frac{x-2a}{2b+x} + \frac{4ab}{x^2-4b^2}.$$

$$40. \frac{1}{a-b} - \frac{1}{a+b} + \frac{1}{a-2b} - \frac{1}{a+2b}.$$

SUGGESTION. — In certain cases, such as this, it is best to combine only part of the fractions at a time. It avoids long multiplications. Here, combine the first and second fractions, then the third and fourth, then the two results obtained.

$$41. \frac{1}{n+1} + \frac{1}{n+3} - \frac{1}{n-1} - \frac{1}{n-3}.$$

$$42. \frac{1}{R-r} - \frac{1}{R+r} + \frac{1}{S-s} - \frac{1}{S+s}.$$

$$43. \frac{2}{y-5} + \frac{3}{y+1} - \frac{2}{y+5} - \frac{3}{y-1}.$$

$$44. \frac{1}{t^2-1} - \frac{1}{t^2+1} + \frac{1}{t^2-a^2} - \frac{1}{t^2+a^2}.$$

$$45. \frac{2A+1}{A^4-1} - \frac{A-3}{A^4+A^2} + \frac{3A-2}{A^4-A^2}.$$

104. Reduction of Mixed Expressions to Fractions.—A mixed expression may be changed to a fraction by the process of § 103. Compare the process as illustrated in the following example to that of reducing an arithmetical mixed number to an improper fraction.

EXAMPLE.—Reduce $m + n + \frac{n^2}{m-n}$ to a fraction.

Writing the integral part as a fraction with the denominator 1,

$$\begin{aligned} m + n + \frac{n^2}{m-n} &= \frac{m+n}{1} + \frac{n^2}{m-n} \\ &= \frac{m^2-n^2}{m-n} + \frac{n^2}{m-n} \\ &= \frac{m^2}{m-n}. \end{aligned}$$

To reduce a mixed expression to a fraction, write the integral part as a fraction with the denominator 1, and proceed as in addition and subtraction of fractions.

EXERCISES

Reduce to a fraction :

1. $1 + \frac{1}{x}$.

4. $\frac{T}{T+2} + 2$.

7. $\pi + \frac{\pi R^2}{H^2}$.

2. $B + \frac{B}{1+B}$.

5. $\frac{m^2}{m-n} - m$.

8. $r_1 - \frac{r_1 r_2}{r_1 + r_2}$.

3. $r + 1 - \frac{r^2}{r-1}$.

6. $w + 5 + \frac{1-2w}{w-3}$.

9. $\frac{2ab}{a-b} + a - b$.

10. $\frac{B^2}{4A} - AC$.

14. $V^2 - \frac{V^2 - 3V^4}{1+V+V^2} - V + 1$.

11. $1 - y + \frac{2y^2}{1+y} - \frac{1}{1-y}$.

15. $4 + \frac{u^2 - 2v^2}{u^2 - uv + v^2} - \frac{u^2v + uv^2}{u^3 + v^3} - 3$.

12. $5 + \frac{t}{1-t^2} + \frac{2t}{t^2-1}$.

16. $\frac{p^2}{(p+q)^2} + 2 + \frac{p}{p+q}$.

13. $a^2 + b^2 - \frac{a^3 - b^3}{a+b} - ab$.

17. $1 + \frac{b^2}{B^2} + \frac{b}{B}$.

105. Multiplication of Fractions.—The product of two or more algebraic fractions is obtained by the same process as the product of two arithmetical fractions.

$$\text{Thus, } \frac{2}{3} \times \frac{7}{9} = \frac{2 \times 7}{3 \times 9} = \frac{14}{27}; \quad \frac{3}{5} \times \frac{2}{7} \times \frac{9}{11} = \frac{3 \times 2 \times 9}{5 \times 7 \times 11} = \frac{54}{385}.$$

$$\text{Similarly, } \frac{a}{b} \times \frac{m}{n} = \frac{am}{bn}; \quad \frac{x+2}{x+5} \times \frac{3x-1}{x-4} = \frac{(x+2)(3x-1)}{(x+5)(x-4)}.$$

The product of two or more fractions is the fraction whose numerator is the product of their numerators and denominator the product of their denominators.

As in arithmetical fractions, the product often may be reduced to lower terms. In such a case, work may be saved by factoring

the terms of the fractions, and cancelling any factor of any numerator by a like factor of any denominator, before performing the multiplications.

EXAMPLE.—Find the product of $\frac{2a^2 + ab - b^2}{2a^2 - ab - b^2}$ and $\frac{3a^2 - ab - 2b^2}{2a^2 + 5ab - 3b^2}$.

$$\begin{aligned} \frac{2a^2 + ab - b^2}{2a^2 - ab - b^2} \times \frac{3a^2 - ab - 2b^2}{2a^2 + 5ab - 3b^2} &= \frac{(a+b)(\cancel{2a-b})}{(2a+b)(\cancel{a-b})} \times \frac{(\cancel{a-b})(3a+2b)}{(\cancel{2a-b})(a+3b)} \\ &= \frac{(a+b)(3a+2b)}{(2a+b)(a+3b)} \\ &= \frac{3a^2 + 5ab + 2b^2}{2a^2 + 7ab + 3b^2}. \end{aligned}$$

EXERCISES

Find the product of:

1. $\frac{5}{6} \times \frac{3}{7}$.
2. $\frac{4}{9} \times \frac{3}{8}$.
3. $\frac{1}{6} \times \frac{12}{17}$.
4. $\frac{1}{3} \times \frac{3}{4} \times \frac{6}{7}$.
5. $\frac{5}{8} \times \frac{16}{21} \times \frac{3}{5}$.
6. $\frac{24}{35} \times \frac{1}{12} \times \frac{5}{6}$.
7. $\frac{1}{4} \times \frac{3}{8} \times \frac{32}{9}$.
8. $\frac{5}{6} \times \frac{7}{8} \times \frac{12}{15}$.
9. $\frac{1}{2} \times \frac{6}{15} \times \frac{20}{21}$.
10. $\frac{A}{B} \times \frac{C}{D}$.
11. $\frac{w^2}{v^2} \times \frac{v}{w}$.
12. $\frac{2x}{3y} \times \frac{9y^2}{10x^2}$.
13. $\frac{n^3}{4} \times \frac{n^5}{5}$.
14. $\frac{3pq}{8r} \times \frac{12r^2}{q}$.
15. $\frac{\pi R^2}{4} \times \frac{2\pi R}{3}$.
16. $\frac{2}{c} \times \frac{c}{2}$.
17. $\frac{5v}{t} \times \frac{n}{v}$.
18. $\frac{7}{15x^3} \times \frac{20x}{21}$.
19. $\frac{a}{3d} \times \frac{6d}{ay}$.
20. $\frac{x}{z^2} \times \frac{y^2}{xz}$.
21. $\frac{a^2}{b^2} \times \frac{bc}{cd} \times \frac{d^2}{a^2}$.
22. $\frac{1}{W} \times \frac{Wg^2}{4} \times \frac{2t^2}{g}$.
23. $\frac{1}{a^2} \times \frac{ab}{a+b}$.
24. $\frac{M}{N} \times \frac{M-N}{5MN}$.
25. $\frac{x+1}{x+2} \times \frac{x+2}{x^2-1}$.
26. $\frac{a^2-b^2}{a^2+b^2} \times \frac{a+b}{a-b} \times \frac{1}{a+b}$.
27. $\frac{m+n}{m^2-n^2} \times \frac{m-n}{m+2n}$.

$$28. \frac{2t-1}{s+3} \times \frac{2s+3}{t-1} \times \frac{s+3}{2t-1} \quad 30. \frac{x-y}{x+2y} \times \frac{x^2-4y^2}{x^3-y^3}.$$

$$29. \frac{A^2-1}{A^2+A+1} \times \frac{A^3-1}{A-1} \quad 31. \frac{v^2+4v}{v+3} \times \frac{v^3+3v^2}{v+4}.$$

$$32. \frac{x^2+2x-3}{x^2+3x-10} \times \frac{x^2-9x+14}{x^2+7x+12}.$$

$$33. \frac{r^2-5r}{r^2+8r+12} \times \frac{r+2}{r^3+r^2+r}.$$

$$34. \frac{\pi R^2 - \pi RH}{R+H} \times \frac{1}{\pi R^2 - \pi H^2}.$$

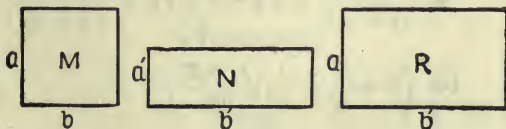
$$35. \frac{6m^2-5mn-4n^2}{4m^2-17mn-15n^2} \times \frac{4m^2-mn-3n^2}{6m^2+7mn+2n^2}.$$

$$36. \frac{V^2+6V+9}{16-8V+V^2} \times \frac{V-4}{9-V^2}.$$

$$37. \frac{b+c}{a+b} \times \frac{a^2+ab+ac+bc}{b^2+bc+bd+cd}.$$

$$38. \frac{1-y^3}{y+1} \times \frac{y^2+3y+2}{y^2+y+1} \times \frac{1}{y-1}.$$

39. M , N , and R are the areas of the rectangles with the dimensions shown in the figures. It is known from Geometry that $\frac{M}{R} = \frac{b}{b'}$ and that $\frac{R}{N} = \frac{a}{a'}$.



Show by multiplication that $\frac{M}{N} = \frac{ab}{a'b'}$.

106. Multiplication of Mixed and Fractional Expressions. — Mixed and fractional expressions to be multiplied should first be reduced to fractions.

EXAMPLE. — Multiply $\frac{9n^2}{v^2} - 1$ by $\frac{v}{3n-v} + 1$.

$$\begin{aligned} \left(\frac{9n^2}{v^2} - 1\right) \left(\frac{v}{3n-v} + 1\right) &= \frac{9n^2 - v^2}{v^2} \times \frac{3n}{3n-v} \\ &= \frac{(3n-v)(3n+v)}{v^2} \times \frac{3n}{3n-v} \\ &= \frac{9n^2 + 3nv}{v^2}. \end{aligned}$$

EXERCISES

Find the product of:

1. $\left(1 + \frac{a}{b}\right) \left(\frac{b^2}{b^2 - a^2}\right).$

4. $\left(r_1 - \frac{r_2^2}{r_1}\right) \left(\frac{r_1}{r_2} + \frac{r_2}{r_1}\right).$

2. $\left(1 - \frac{R^2}{D^2}\right) \left(\frac{1}{R-D}\right).$

5. $\left(\frac{9h^2}{b^2} - 1\right) \left(\frac{b}{3h-b} + 1\right).$

3. $\left(\frac{uv}{u^2 + v^2}\right) \left(1 - \frac{u^4}{v^4}\right).$

6. $\left(M + \frac{1}{N}\right) \left(M - \frac{1}{N}\right).$

7. $\left(\frac{1}{x} - 1\right) \left(\frac{1}{x} - 1\right).$

8. $\left(\frac{y^2 + 2yz + z^2}{yz} - 4\right) \left(1 + \frac{4yz}{y^2 - 2yz + z^2}\right).$

9. $\left(\frac{36}{2k+3} + 2k - 9\right) \left(2k + \frac{6k-9}{2k-3}\right).$

10. $\left(1 + \frac{1}{E} + \frac{1}{E^2}\right) \left(\frac{8E^3}{E^3 - 1}\right).$

11. $\left(A + \frac{AB}{A+B}\right) \left(A - \frac{AB}{A+B}\right) \left(\frac{A^2 - B^2}{A^2 + B^2}\right).$

$$12. \left(\frac{1}{t^2} - 1\right) \left(\frac{t}{1+t}\right) \left(\frac{1-t^2}{1-2t+t^2}\right).$$

$$13. \left(\frac{P}{1+P} + \frac{1-P}{P}\right) \left(\frac{P}{1+P} - \frac{1-P}{P}\right).$$

$$14. \frac{s}{2} \left(\frac{s^2-t^2}{s^2t+st^2}\right) \left(\frac{1}{s-t} - \frac{1}{s+t}\right).$$

$$15. \left(\frac{1}{p-q} - \frac{1}{p+q} + \frac{2}{p}\right) \left(\frac{p^2-q^2}{4pq}\right).$$

107. Multiplication of Integral Expressions and Fractions.—In the multiplication of a fraction by an integral expression, the latter may first be written as a fraction with the denominator 1, and then the product found as in the multiplication of fractions.

Thus, $12 \times \frac{3}{56} = \frac{12}{1} \times \frac{3}{56} = \frac{9}{14}$.

And $(n^2-4) \left(\frac{n^2+4n-5}{n^2-n-6}\right) = \frac{n^2-4}{1} \times \frac{n^2+4n-5}{n^2-n-6} = \text{etc.}$

It frequently is found necessary, as in solving equations containing fractions, to multiply a fractional expression by the L. C. D. of the fractions involved.

EXAMPLE.—Multiply $\frac{2a^2}{a^2-1} + \frac{a}{a-1} - \frac{a}{a+1}$ by the lowest common denominator of all three fractions.

The L. C. D. is a^2-1 . Multiplying by this gives

$$\begin{aligned} \frac{a^2-1}{1} \times \frac{2a^2}{a^2-1} + \frac{a^2-1}{1} \times \frac{a}{a-1} - \frac{a^2-1}{1} \times \frac{a}{a+1} \\ = 2a^2 + a^2 + a - a^2 + a \\ = 2a^2 + 2a. \end{aligned}$$

EXERCISES

Multiply:

$$1. 3 \times \frac{5}{6}.$$

$$3. 9 \times \frac{11}{12}.$$

$$5. 15 \times \frac{32}{36}.$$

$$2. 14 \times \frac{16}{21}.$$

$$4. 24 \times \frac{5}{16}.$$

$$6. A^2 \times \frac{B}{A}.$$

7. $z^3 \times \frac{3xy}{z^2}.$

8. $5w \times \frac{3v}{20w}.$

9. $\pi R^2 \times \frac{R+H}{2\pi R}.$

10. $(n^2-9)\left(\frac{n^2+4n-12}{n^2-4n+3}\right).$

12. $(t^3-1)\left(\frac{2t}{t^2-7t+6}\right).$

11. $(\pi R^2 - \pi H^2)\left(\frac{R+H}{R-H}\right).$

13. $(2v^2-8)\left(\frac{v}{v+2}\right).$

14. $(2x^2+xy-6y^2)\left(\frac{x^2}{4x^2+9xy+2y^2}\right).$

15. $(r_1+r_2)\left(\frac{r_1r_2}{r_1^2-2r_1r_2-3r_2^2}\right).$

Multiply by the lowest common denominator:

16. $\frac{1}{a+2} + \frac{1}{a-2}.$

20. $\frac{g}{3} - \frac{g^2-5g}{3g-7} - \frac{2}{3}.$

17. $\frac{2y}{y-3} + \frac{3y}{y+3} - \frac{y^2}{y^2-9}.$

21. $\frac{2D+4}{3D-5} - \frac{5}{2}.$

18. $\frac{1}{m+1} - \frac{1}{m-1} + \frac{1}{m+2}.$

22. $\frac{3R+5}{4R-3} - \frac{3R^2+5R-4}{4R^2-3R+2}.$

19. $\frac{8}{P} + \frac{P-3}{P+3} - \frac{P+1}{P-1}.$

23. $\frac{2k+3}{3k-4} - \frac{4k+5}{6k-1}.$

24. $\frac{x+y}{x^2+xy+y^2} + \frac{1}{x-y} - \frac{xy+2y^2}{x^3-y^3}.$

25. $\frac{2M}{2M+3} - \frac{1-M}{2-M} + \frac{7}{2M^2-M-6}.$

108. Division of Fractions.—The quotient of two algebraic fractions is obtained by the same process as the quotient of two arithmetical fractions; i.e.,

To find the quotient of two algebraic fractions, multiply the dividend by the divisor inverted.

This follows from the principle that the product of the quotient and divisor must equal the dividend. If $\frac{a}{b}$ and $\frac{c}{d}$ represent any two fractions, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$, because $\frac{a}{b} \times \frac{d}{c}$ multiplied by $\frac{c}{d}$ gives $\frac{a}{b}$ by cancellation.

EXAMPLE.—Divide $\frac{A^2 - 4AB + 3B^2}{A^2 + 2AB + B^2}$ by $\frac{A - 3B}{A + B}$.

$$\begin{aligned} \frac{A^2 - 4AB + 3B^2}{A^2 + 2AB + B^2} \div \frac{A - 3B}{A + B} &= \frac{(A - 3B)(A - B)}{(A + B)^2} \times \frac{A + B}{A - 3B} \\ &= \frac{A - B}{A + B}. \end{aligned}$$

NOTE.—If a dividend or divisor is a mixed or fractional expression, it may be changed to a fraction, and the division performed by the above process.

EXERCISES

Perform the indicated divisions:

1. $\frac{2}{3} \div \frac{5}{6}$.
2. $\frac{7}{8} \div \frac{3}{4}$.
3. $\frac{2}{9} \div \frac{4}{5}$.
4. $\frac{11}{12} \div \frac{1}{6}$.
5. $\frac{15}{16} \div \frac{3}{8}$.
6. $\frac{7}{24} \div \frac{5}{6}$.
7. $\frac{3}{17} \div \frac{2}{5}$.
8. $\frac{81}{100} \div \frac{3}{25}$.
9. $\frac{a}{2} \div \frac{3a}{4m}$.
10. $\frac{3\pi R}{4D^2} \div \frac{6R^2}{2D^2}$.
11. $\frac{4v^2}{9t} \div \frac{2v}{3}$.
12. $\frac{8W}{9P} \div \frac{W}{3}$.
13. $\frac{3x^2y}{5z^2} \div \frac{2xy^2}{10zw}$.
14. $\frac{20H^2}{21L^2} \div \frac{4HL}{3H^2}$.
15. $\frac{36r_1^4r_2^2}{35R_1^3R_2^2} \div \frac{12r_1^2r_2}{7R_1^2R_2}$.
16. $\frac{a+1}{a} \div \frac{a^2-1}{a^2}$.
17. $\frac{6M^2-2M}{4-M^2} \div \frac{M^2}{2+M}$.
18. $\frac{W^2-V^2}{W+2V} \div \frac{W-V}{3W+6V}$.
19. $\frac{x^2-18x+80}{x^2-5x-50} \div \frac{x^2-15x+56}{x^2-6x-7}$.
20. $\frac{R^2-1}{R^2-3R-10} \div \frac{R^2-12R+35}{R^2+3R+2}$.

$$21. \frac{2t^2}{a^3+t^3} \div \frac{t}{a+t}.$$

$$26. \frac{n-1}{3n^2+8n+5} \div \frac{n^3-1}{n^2+4n+3}.$$

$$22. \frac{b^2+c^2}{b^2-c^2} \div \frac{3b^2+3c^2}{b+c}.$$

$$27. \frac{x^2-9y^2}{x^2+4y^2} \div \frac{x^2+xy-6y^2}{x^2-xy-6y^2}.$$

$$23. \frac{2mn-n^2}{mn+m^2} \div \frac{n^2-m^2}{n+m}.$$

$$28. \frac{p^3-q^3}{p^2-pq+q^2} \div \frac{p^2+pq+q^2}{p^3+q^3}.$$

$$24. \frac{A^2-B^2}{A} \div (A-B).$$

$$29. (t-3v) \div \frac{2t^2-5tv-3v^2}{t^2}.$$

$$25. \left(1 - \frac{a^2}{b^2}\right) \div \frac{a+b}{ab}.$$

$$30. \left(\frac{V^2}{t^2} + \frac{2V}{t} + 1\right) \div \left(1 + \frac{V^3}{t^3}\right).$$

109. Complex Fractions.—Fractions whose numerators, denominators, or both, contain fractions are called **complex fractions**.

Thus, $\frac{\frac{3}{5}}{\frac{2}{3}}$, $\frac{\frac{a}{b}}{\frac{c}{d}}$, $\frac{W}{\frac{W+V}{WV}}$, and $\frac{1+\frac{1}{t}}{1-\frac{1}{t}}$ are complex fractions.

A complex fraction is said to be *simplified* when it is reduced to an equivalent fraction whose terms are integral expressions, or to an integral expression. Since a complex fraction may be considered an indicated quotient, it may be simplified by dividing the numerator by the denominator.

EXAMPLE.—Simplify $\frac{A - \frac{1}{A}}{1 + \frac{1}{A}}$.

$$\frac{A - \frac{1}{A}}{1 + \frac{1}{A}} = \frac{\frac{A^2-1}{A}}{\frac{A+1}{A}} = \frac{A^2-1}{A} \times \frac{A}{A+1} = A-1.$$

EXERCISES

Simplify:

1. $\frac{5}{\frac{2}{3}}$
2. $\frac{\frac{4}{2}}{\frac{2}{5}}$
3. $\frac{2\frac{3}{4}}{3\frac{2}{3}}$
4. $\frac{8\frac{1}{3}}{\frac{5}{6}}$
5. $\frac{\frac{A}{B}}{\frac{A^2}{B^2}}$
6. $\frac{x^2}{\frac{1}{x}}$
7. $\frac{w+1}{1+\frac{1}{w}}$
8. $\frac{\frac{m+n}{x}}{\frac{m^2-n^2}{x^2}}$
9. $\frac{\frac{y}{ay+b}}{a}$
10. $\frac{\frac{1}{R-H}}{\frac{1}{R+H}}$
11. $\frac{\frac{u}{u-v}}{\frac{u^2}{u^2-v^2}}$
12. $\frac{1+\frac{M}{N}}{\frac{M}{N}-1}$
13. $\frac{\frac{c+2}{c^2-4}}{2}$
14. $\frac{\frac{1-r_1}{r_2}}{r_2^2-r_1^2}$
15. $\frac{\frac{H+R}{\pi}}{\frac{R^2}{H^2}}$
16. $\frac{\frac{x}{1+x}+1}{\frac{1}{1+x}+x}$
17. $\frac{A-1+\frac{6}{A-6}}{A-2+\frac{3}{A-6}}$
18. $\frac{y+\frac{y}{z}}{1+\frac{1}{z}}$
19. $\frac{\frac{x-y}{x}-\frac{x+y}{y}}{\frac{x-y}{y}+\frac{x+y}{x}}$
20. $\frac{\frac{R}{1+R}+\frac{1-R}{R}}{\frac{R}{1+R}-\frac{1-R}{R}}$
21. $\frac{\frac{r_1^2}{r_2^2}-\frac{r_2^2}{r_1^2}}{r_1+\frac{r_2}{r_1}}$

SUPPLEMENTARY EXERCISES

NOTE.—In some problems in the addition or subtraction of fractions, such as the following, it is advisable to change the signs of factors of one or more denominators before finding the lowest common denominator. In this change of signs use is made of the principle that, *if the sign of one factor of a product is changed, the sign of the product is changed; and if the signs of two factors of a product are changed, the sign of the product is not changed.*

Thus, $(a-2)(a-4)=a^2-6a+8$. If the sign of $a-2$ is changed, by changing the signs of its terms, we get $(2-a)(a-4)=-a^2+6a-8$. If the signs of both factors are changed, we get $(2-a)(4-a)=a^2-6a+8$.

Perform the indicated additions and subtractions:

$$1. \frac{1}{(A-B)(A-C)} + \frac{1}{(B-C)(B-A)} + \frac{1}{(C-A)(C-B)}.$$

SOLUTION. — Writing all factors of denominators as $A-B$, $B-C$, or $C-A$,

$$\begin{aligned} & \frac{1}{(A-B)(A-C)} + \frac{1}{(B-C)(B-A)} + \frac{1}{(C-A)(C-B)} \\ &= \frac{-1}{(A-B)(C-A)} + \frac{-1}{(B-C)(A-B)} + \frac{-1}{(C-A)(B-C)} \\ &= \frac{-1(B-C)}{(A-B)(B-C)(C-A)} + \frac{-1(C-A)}{(A-B)(B-C)(C-A)} \\ & \quad + \frac{-1(A-B)}{(A-B)(B-C)(C-A)} \\ &= \frac{-B+C-C+A-A+B}{(A-B)(B-C)(C-A)} = 0. \end{aligned}$$

$$2. \frac{1}{(x-y)(x-z)} - \frac{1}{(y-x)(y-z)} + \frac{1}{(x-z)(z-y)}.$$

$$3. \frac{mn}{(p-m)(p-n)} + \frac{np}{(m-n)(m-p)} + \frac{pm}{(n-p)(n-m)}.$$

$$4. \frac{u}{(v-w)(w-u)} + \frac{v}{(w-u)(u-v)} + \frac{w}{(u-v)(v-w)}.$$

$$5. \frac{r_1+r_2}{(r_2-r_3)(r_3-r_1)} + \frac{r_2+r_3}{(r_3-r_1)(r_2-r_1)} + \frac{r_3+r_1}{(r_2-r_1)(r_3-r_2)}.$$

$$6. \frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-c)(b-a)} + \frac{c^2}{(c-a)(c-b)}.$$

$$7. \frac{u(v+w)}{(u-v)(w-u)} + \frac{v(w+u)}{(v-w)(u-v)} + \frac{w(u+v)}{(w-u)(v-w)}.$$

$$8. \frac{1}{(x-1)(x-2)} + \frac{1}{(2-x)(3-x)} + \frac{1}{(x-1)(3-x)}.$$

$$9. \frac{6}{a^2-4a+3} + \frac{3}{7a-6-a^2} + \frac{1}{9a-a^2-18}.$$

NOTE. — In the following problems invert divisors, and find the products of all factors at one step by cancellation.

Simplify:

$$10. \frac{x^2}{y^2} \times \frac{3y}{5x} \div \frac{9x}{10y}.$$

$$11. \frac{a}{b} \div \frac{a^2}{b^2} \div \frac{a^3}{b^3}.$$

$$12. \frac{n-1}{n-2} \times \frac{n^2+3n-10}{n^2+n-2} \div \frac{n^2+5n}{n^2-n-6}$$

$$13. (3u-2v) \left(2 - \frac{u-2v}{2u+v} \right) \div \frac{9u^2+6uv-8v^2}{2u+v}.$$

$$14. \frac{t^2-16}{t^2-5t+6} \times \frac{t^2-3t-10}{t^2-11t+30} \div \frac{t^2-9t-52}{t^2+5t-66}.$$

$$15. \frac{6P^2-11P-10}{3P^2+2P-5} \div \frac{7P^2+17P-12}{5P^2+9P-2} \div \frac{10P^2-27P+5}{21P^2+23P-20}.$$

$$16. \left(1 + \frac{3}{a} + \frac{2}{a^2} \right) \div \left(1 + \frac{1}{a^3} \right) \div (a+2)^2.$$

$$17. \left(4 - \frac{16N}{M} + \frac{15N^2}{M^2} \right) \div \left(\frac{2M-5N}{2M+5N} \right) \div \left(\frac{4N}{M} + 4 - \frac{15N^2}{M^2} \right).$$

NOTE. — A good method of simplifying a complex fraction (see § 109) consists of multiplying both terms of the fraction by the lowest common denominator of all simple fractions in the terms, as suggested in the following problem.

$$18. \frac{\frac{a+b}{a} + \frac{a-b}{b}}{\frac{a-b}{a} - \frac{a+b}{b}}.$$

SUGGESTION. — Multiplying both terms by the L. C. D. gives

$$\frac{b(a+b) + a(a-b)}{b(a-b) - a(a+b)} = \text{etc.}$$

$$19. \frac{\frac{1}{n+2} + \frac{2}{n+3}}{\frac{3}{n+2}}.$$

$$21. \frac{\frac{3x}{2y} + 2 + \frac{2y}{3x}}{\frac{1}{2y} + \frac{1}{3x}}.$$

$$20. \frac{\frac{n}{y^2}}{1 + \frac{n^2}{y^2}}.$$

$$22. \frac{\frac{2A^2 - AB - 3B^2}{3A^2 - AB - 2B^2}}{\frac{A^2 + 3AB + 2B^2}{4A^2 - 5AB + B^2}}.$$

CHAPTER X

FRACTIONAL EQUATIONS. PROBLEMS. FORMULÆ

110. Fractional Equations. — Some problems are expressed and solved by means of equations in which the unknown number appears in one or more denominators of fractions. An equation of which the unknown number appears in one or more denominators is called a **fractional equation**.

Thus, $\frac{n+1}{n-2} = \frac{n}{n+2} + 5$ is a fractional equation.

111. Clearing of Fractions. — To solve a fractional equation it is necessary first to change it so that it is free of fractions. This process is called **clearing the equation of fractions**.

It was found in § 107 that if a mixed or fractional expression is multiplied by the lowest common denominator of all of the fractions involved, the product is integral. We know also, as an axiom, that if equal quantities, such as the members of an equation, are multiplied by the same quantity, the products are equal. These principles are used in clearing an equation of fractions.

EXAMPLE. — Solve $\frac{2t^2}{t^2-1} + \frac{t}{t-1} = \frac{t}{t+1} + 2$.

The L. C. D. of all fractions in the equation is $t^2 - 1$. Multiplying both members of the equation by $t^2 - 1$, by multiplying every term of each member, gives

$$(t^2 - 1) \left(\frac{2t^2}{t^2 - 1} \right) + (t^2 - 1) \left(\frac{t}{t - 1} \right) = (t^2 - 1) \left(\frac{t}{t + 1} \right) + 2(t^2 - 1),$$

or

$$2t^2 + t^2 + t = t^2 - t + 2t^2 - 2.$$

Solving,

$$2t = -2,$$

$$t = -1.$$

To clear an equation of fractions, multiply each member, by multiplying every term of each member, by the lowest common denominator of all fractions in the equation.*

EXERCISES

Solve:

1. $2 = \frac{6}{n}$.
2. $\frac{3}{2y} = \frac{1}{4}$.
3. $\frac{4}{A} = \frac{5}{A} - 1$.
4. $\frac{1}{r} + \frac{2}{r} + \frac{3}{r} = 1$.
5. $\frac{x+1}{x^2} + \frac{2}{3x} = \frac{1}{x}$.
6. $\frac{4}{5c} - \frac{7}{10c} = \frac{1}{10}$.
7. $\frac{42}{T} - \frac{1}{T} + \frac{1}{3T} = 20$.
8. $\frac{m+12}{2m} = \frac{2m+12}{3m}$.
9. $\frac{2}{1-z} = 1$.
10. $\frac{2}{v} = \frac{1}{v-1}$.
11. $\frac{S+3}{S-3} = 3$.
12. $\frac{3}{k+1} = \frac{2}{k-1}$.
13. $\frac{6}{2t+3} = \frac{2}{t-4}$.
14. $\frac{n-3}{n+9} = \frac{n-5}{n+5}$.
15. $\frac{D-1}{2D-5} = \frac{D-5}{2D-2}$.
16. $\frac{9}{5r+2} - \frac{7}{3r+4} = 0$.
17. $\frac{B}{B-2} + \frac{6}{B+2} = 1$.
18. $1 = \frac{y}{y-2} + \frac{4}{y+1}$.
19. $\frac{2P+3}{3P-4} - \frac{4P+5}{6P-1} = 0$.
20. $\frac{w}{w+1} = \frac{3w}{w+2} - 2$.

* It is important that the teacher should see that no pupil gets a wrong idea of the process of clearing of fractions through careless thinking. Thus, a pupil sometimes thinks that clearing an equation such as $\frac{x-1}{2x+4} = \frac{3-x}{1+x}$ of fractions consists of multiplying the first member by $1+x$ and the second by $2x+4$, rather than each member by $(1+x)(2x+4)$.

$$21. \frac{4}{A-5} = \frac{5}{A}.$$

$$24. \frac{3}{N+1} - \frac{2}{N-1} = 0.$$

$$22. \frac{t+1}{t^2} + \frac{2}{3t} = \frac{1}{t}.$$

$$25. \frac{c+3}{c} = \frac{c+9}{c+4}.$$

$$23. \frac{15y^2 - 5y - 8}{3y^2 + 6y + 4} = 5.$$

$$26. \frac{2n+1}{4} - \frac{1}{n-1} = \frac{n}{2}.$$

$$27. \frac{1}{2i+3} + \frac{1}{2i-3} = \frac{1}{4i^2-9}.$$

$$28. \frac{5}{z+1} = \frac{10-7z}{z-1} + 7.$$

$$29. \frac{3}{m+1} - \frac{2}{m-2} = \frac{1}{m+2}.$$

$$30. \frac{6}{V+2} + \frac{2}{V-2} = \frac{16}{V^2-4}.$$

$$31. \frac{8}{p} + \frac{p-3}{p+3} = \frac{p+1}{p-1}.$$

$$32. \frac{3a+5}{4a-3} = \frac{3a^2+5a-4}{4a^2-3a+2}.$$

$$33. \frac{2x+1}{2x-1} - \frac{8}{4x^2-1} = \frac{2x-1}{1+2x}.$$

$$34. \frac{v^2-3}{1-v^2} = \frac{4}{1+v} + \frac{v+1}{1-v}.$$

$$35. \frac{37}{R^2+5R+6} - \frac{4}{R+2} = \frac{7}{R+3}.$$

$$36. \frac{1}{T-3} = \frac{T-6}{T+3} - \frac{T}{T+3}.$$

$$37. \frac{7}{4L-12} + \frac{47}{220} + \frac{3}{5L-15} = 0.$$

$$38. \frac{x-5}{x^2-10x+21} - \frac{7-x}{x^2-8x+15} = \frac{2x-15}{x^2-12x+35}$$

$$39. \frac{9A+17}{A^2-2A-48} + \frac{2A-1}{2A+12} = \frac{2A+1}{2A-16}$$

$$40. \frac{t^2-2t+5}{t^2-2t-3} + \frac{t^2+3t-7}{t^2+3t+2} = 2.$$

$$41. \frac{N-4}{N-5} + \frac{N-15}{N+4} = \frac{2N^2-10N-1}{N^2-N-20}.$$

$$42. \frac{7a^2+11a+4}{6a^2+13a+5} + \frac{a+3}{2a+1} = \frac{5a+11}{3a+5}.$$

112. Problems Leading to Fractional Equations. — Types of problems such as those in the following list lead to fractional equations, which are solved by the method of § 111.

EXERCISES

1. The sum of two numbers is 265, and if the larger be divided by the smaller, the quotient is 14 and remainder 10. Find the numbers.

SUGGESTION. — Let n = the smaller number.

Then $265 - n$ = the larger number.

Hence,
$$\frac{265-n}{n} = 14 + \frac{10}{n}.$$

2. The sum of two numbers is 160, and if the larger be divided by the smaller, the quotient is 4. Find the numbers.

3. The difference of two numbers is 324, and if the larger be divided by the smaller, the quotient is 3 and remainder 80. Find the numbers.

4. Separate 84 into two parts whose quotient is $\frac{3}{4}$.

5. Separate 148 into two parts such that if the larger be divided by the smaller, the quotient is 5 and remainder 16.

6. Find where to divide a beam 18 ft. long into two parts such that their quotient is $3\frac{1}{2}$.

7. A board is 42 in. long. A hole is to be bored in the board so that its distance from one end divided by its distance from the other end shall be $\frac{3}{4}$. How far from one end must the hole be bored?

8. A surveyor has two stakes set in the ground 150 ft. apart. He wishes to drive a third stake between them such that its distance to the nearer of the two divided by its distance to the other shall be $\frac{2}{3}$. Where must he set the third stake?

9. What number must be subtracted from each term of the fraction $\frac{10}{13}$ in order that the result shall be equal to $\frac{1}{2}$?

10. What number must be subtracted from each term of the fraction $\frac{17}{13}$ in order that the result shall be equal to $\frac{3}{8}$?

11. What number must be subtracted from each term of the fraction $\frac{21}{26}$ in order that the result shall be equal to $\frac{3}{4}$?

12. What number must be added to each term of the fraction $\frac{13}{25}$ so that the result shall be equal to $\frac{5}{9}$?

13. What number must be added to each term of the fraction $\frac{19}{186}$ so that the result shall be equal to $\frac{3}{16}$?

14. What number must be subtracted from each of the numbers, 14, 20, 32, and 50, so that the quotient of the first two remainders shall be equal to the quotient of the last two remainders?

15. In a certain factory a large machine can turn out a number of articles in 4 days, and a smaller sized machine can turn them out in 6 days. If both machines are operated at once, how long will it take them to turn out the articles?

SUGGESTION.—Let n = number of days required for the machines together to turn out the articles.

Show that

$$\frac{1}{n} = \frac{1}{4} + \frac{1}{6}.$$

16. In a factory which manufactures a certain kind of goods there are 3 large machines and 5 small ones. To manufacture the goods required to fill an order would take either one of the

large machines alone 60 days, and either of the small machines alone 90 days. If all 8 machines are operated, how long does it take to fill the order?

17. In a city water system, two pumps pump the water into a reservoir. One pump would fill the reservoir in 16 hr., and the other would fill it in 12 hr. If both pumps were driven at once, how long would it require to fill the reservoir?

18. A tank is fitted with two pipes. One pipe alone could fill the tank in 5 hr., and the other pipe alone in 8 hr. If both pipes were opened at once, how long would it require to fill the tank?

19. A tank can be filled by one pipe in 4 hr. and emptied by another in 5 hr. If both pipes are open, how long will it take to fill the tank?

20. A railroad water tank can be pumped full in $4\frac{1}{2}$ hr. But locomotives draw out and use, on the average, a tankful every 8 hr. At this rate, how long would it take to get the tank pumped full?

21. Of two furnaces in a school building one will burn a given quantity of coal in 8 days and the other will burn it in 10 days. If both furnaces are fired, how long will the quantity of coal last?

22. A farmer has a quantity of corn which he feeds to his horses and hogs. It would last the horses alone 160 days, and the hogs alone 60 days. How long will it last if he feeds it to both horses and hogs?

23. Assuming that they work at constant speeds, A can do a piece of work in 7 days that it would take B 9 days to do. If they work together, how long should it require them both to do the work?

24. A can do a piece of work in 10 days; but after he has worked 2 days, B comes to help him, and together they finish it in 3 more days. In how many days could B alone have done the work?

25. John could remove the snow from the sidewalk in 30 minutes. His larger brother James could do it in 20 minutes

John began the work, but later James took his place, and the snow was all removed in 25 minutes from the beginning. How long did John work?

26. A steamboat makes 8 miles an hour against the wind on a journey of 40 miles. After it has gone 15 miles, the wind ceases. The entire time consumed on the journey is $4\frac{1}{2}$ hr. Find how many miles per hour the wind retards the boat.

27. A train runs into a city from a suburban town, a distance of 40 mi., in 1 hr. 40 min. Of this distance 8 mi. are within the city limits. The train makes 50 mi. an hour outside of the city limits. What is its speed within the city limits?

28. Two trains, approaching each other, leave stations 225 miles apart at the same moment. One train runs 5 miles an hour faster than the other, and they meet at a point 120 miles from the station from which the faster train starts. Find the speeds of the trains.

29. A motorcyclist was overtaken 75 miles from his starting point by an automobile which started from the same place 2 hours later, and traveled 10 miles an hour faster. Find the speed of each.

30. A messenger was started on a journey with a message that later was found to be wrong. One hour and 20 minutes after he left, a second messenger started to overtake him, traveling 2 miles an hour faster. The first messenger was overtaken after he had gone a distance of 16 miles. Find the speed of each.

31. A man started on a journey of 72 miles. After going 42 miles, he stopped and rested 2 hours. Including the 2 hours that he rested, to finish the journey at the same speed took him as long as to go the first 42 miles. Find how fast he traveled.

32. An aviator flew to a point 60 miles away, and back, in 5 hr. 24 min. His rate going was 25 miles an hour. What was his rate returning?

33. An aviator started on a trip to a point 80 miles away. After going 30 miles he increased his speed 20 miles an hour,

and made the remaining distance in the same time that it took him to fly the first 30 miles. What was his speed the first 30 miles?

34. An aviator made 25 miles an hour against the wind on a journey of 45 miles. After he had gone 20 miles, the wind ceased. The entire time required for the trip was 1 hr. 25½ min. What was the speed of the wind?

35. In the National League of baseball teams, at one point of the season, Chicago and New York had records as follows:

	WON	LOST
Chicago	64	30
New York	58	32

If these two teams were to play a final series of 6 games together, how many would Chicago have to win in order that the quotient of the number of games won and the number lost should be greater for Chicago than for New York?

SUGGESTION. — Let x be the number Chicago must win in order that the quotients of the numbers of games won and the numbers lost should be the same for both teams.

Show that
$$\frac{64 + x}{30 + 6 - x} = \frac{58 + 6 - x}{32 + x}.$$

Show that
$$x = 1\frac{1}{2}.$$

Hence, Chicago must win two more games in order to finish ahead of New York.

36. The Pittsburg and Philadelphia baseball teams of the National League had records as follows:

	WON	LOST
Pittsburg	54	34
Philadelphia	50	36

If these two teams were to play a final series of 5 games together, how many would Pittsburg have to win in order that the quotient of the number of games won and the number lost should be greater for Pittsburg than for Philadelphia?

113. Literal Equations: Formulæ.—An equation that contains two or more letters, such as the practical formulæ encountered in the earlier part of the book, is called a **literal equation**.

In a literal equation the number represented by any one of the letters involved may be considered as the unknown number, and its value solved for. It is clear that in such an equation the value found for the unknown number will, in general, be an expression containing the other letters involved in the equation. If a literal equation is linear with respect to the unknown number, it may be solved by the method of this chapter.

EXAMPLE.—Solve $x = \frac{na + b}{n}$ for n .

Clearing of fractions, $nx = na + b$.

Transposing, $nx - na = b$.

Uniting terms, $(x - a)n = b$.

Dividing by $x - a$, $n = \frac{b}{x - a}$.

NOTE.—The process of uniting the similar terms in a literal equation, as in step 3 of the above example, is equivalent to factoring the terms.

EXERCISES

1. Solve $x = \frac{na + b}{n}$ for a .

2. Solve $2a - \frac{b}{x} = \frac{cy}{x}$ for x .

3. Solve $a = \frac{x - a}{x - 1}$ for a .

4. Solve $2 + \frac{3}{2t - 1} = \frac{v}{2t - 1}$ for t .

5. Solve $\frac{R}{R-2} + \frac{V}{R+1} = 1$ for V .
6. Solve $\frac{y}{m} + \frac{y}{n} = m + n$ for y .
7. Solve $\frac{A}{B} = \frac{t-B^2}{t-A^2}$ for t .
8. Solve $\frac{m+r}{n+r} - \frac{2}{r^2-n^2} = \frac{m-r}{n-r}$ for r .
9. Solve $\frac{1}{P} + \frac{1}{Q} + \frac{1}{R} = 1$ for P .
10. Solve $\frac{1}{N} + 1 - \frac{N}{d} + \frac{1}{dN} = 0$ for d .
11. Solve $\frac{9+3x}{2xy} + \frac{y}{y(y-t)} = \frac{3}{y(t-y)}$ for t .
12. Solve $\frac{3A+4C}{3A} - \frac{4A+3C}{4A} = \frac{1}{10}$ for A .
13. Solve $\frac{2P+2W}{P^2-PW} - \frac{P+W}{P^2-2PW+W^2} = \frac{P+W}{P^2-W^2}$ for P .
14. Solve $\frac{2c^2+d^2}{c^2-d^2} = \frac{c}{c+d} + 1$ for c .

NOTE.—The following formulæ express important principles in science, geometry, etc. It often is necessary, in using such formulæ, to express one of the quantities involved in terms of the others. The pupil will find it of decided advantage to learn that process here.

15. If C and c are the circumferences, and R and r the radii, respectively, of any two circles, then

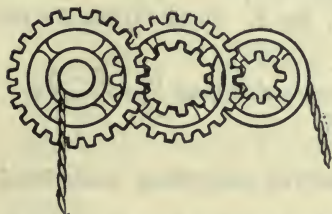
$$\frac{C}{c} = \frac{R}{r}.$$

Solve for r .

16. The figure bounded by two radii of a circle and the arc subtended by them is a sector of the circle. If s and a are the area and arc, respectively, of a sector of a circle whose area is S and circumference C , then $\frac{s}{S} = \frac{a}{C}$.

Solve for s ; for S ; for a ; for C .

17. If one gear or cogwheel drives another, and S and T are the speed and number of teeth of the driving wheel, and s and t the speed and number of teeth of the driven wheel, respectively,



then $\frac{S}{s} = \frac{t}{T}$.

Solve for S ; for s ; for t ; for T .

If the number of teeth of the driving wheel is 96, of the driven wheel 12, and the speed of the driving wheel 40 revolutions a minute, find the speed of the driven wheel.

18. The formula for computing the area L of a lune, the portion of the surface of a sphere bounded by two semicircles such as the meridians on the earth's surface, is

$$\frac{L}{8T} = \frac{A}{4}.$$

Solve for L ; for A .

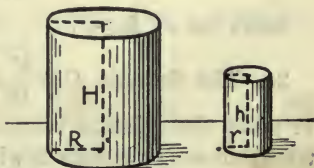
19. In two similar right cylinders whose volumes, altitudes, and radii are V , H , R , and v , h , r , respectively,

$$\frac{V}{v} = \frac{\pi R^2 H}{\pi r^2 h}.$$

Solve for V ; for v ; for H ; for h .

20. If T and t are the areas of the entire surfaces of the above cylinders,

$$\frac{T}{t} = \frac{R(H+R)}{r(h+r)}.$$



Solve for T ; for t ; for H ; for h .

21. If the radius of a right cylinder is denoted by R , the area of the curved surface by S , the area of the total surface by T , the altitude by H , and the volume by V ,

$$(1) R = \frac{V}{S}; \quad (2) RH = \frac{S}{2\pi}; \quad (3) R^2H = \frac{V}{\pi};$$

$$(4) R^2 + HR = \frac{T}{2\pi}; \quad (5) R^2 + HR = \frac{T+S}{2\pi}.$$

Solve (1) for V ; for S . Solve (2) for R ; for H ; for S . Solve (3) for H ; for V . Solve (4) for H ; for T . Solve (5) for T ; for S ; for H .

22. In the above cylinder it is known also that $\frac{H}{R} = \frac{T-S}{S}$.
Solve for H ; for R ; for T ; for S .

23. The formula $F = \frac{Mv^2}{r}$ is used in computing centrifugal force, or the force with which an object moving in a circle tends to fly off from the center. It is the force which causes grindstones and flywheels to burst when run at too high a speed, and carriages to overturn when turning a corner at too high a speed.

Solve for M ; for r .

24. Solve $f = \frac{wv^2}{gr}$ for w ; for g ; for r .

25. The densities of some substances are computed by the formula,

$$D = \frac{w}{w - w'}.$$

Solve for w ; for w' .

26. The formula $C = \frac{E}{R}$ is of great importance in electricity.

It gives the relation between the quantity of current C of electricity flowing through a wire or other conductor, the resistance offered to the current by the conductor, and the electrical pressure (electromotive force) required to overcome that resistance.

Solve for E ; for R .

27. If the resistance within an electric battery cell is R , the resistance of the external circuit through which the current flows r , the strength of current C , and the electric pressure E , then

$$C = \frac{E}{R + r}.$$

Solve for E ; for R ; for r .

28. If n equal electric cells are connected by wires in one way (series), so as to form a battery, the current strength is found by the formula

$$C = \frac{nE}{R + nr}.$$

Solve for n ; for E ; for R ; for r .

29. If the cells in Problem 28 are connected in another way (parallel), the current strength of the battery is found by the formula

$$C = \frac{E}{R + \frac{r}{n}}.$$

Solve for n ; for E ; for R ; for r .

30. If an electric circuit between two points is divided into two branches, so that part of the current flows through one branch and the rest through the other, the total resistance of the circuit is computed by the formula

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2},$$

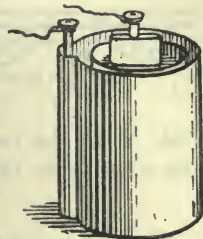
where R is the total resistance, r_1 the resistance of one branch, and r_2 the resistance of the other branch.

Solve for R .

31. If the circuit in Problem 30 is divided into three branches whose resistances are r_1 , r_2 , and r_3 , the formula is

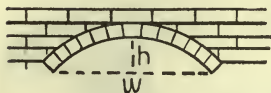
$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

Solve for R .



32. Solve $S = \frac{a - rl}{1 - r}$ for r ; for a ; for l .

33. The radius r of the arch of a bridge or doorway whose span is w and rise h is found from the formula



$$\frac{2r - h}{\frac{1}{2}w} = \frac{\frac{1}{2}w}{h}.$$

Solve for r .

34. Find the radius at which the stones must be cut for an arch whose span is 16 ft. and rise 4 ft.

SUPPLEMENTARY EXERCISES

Solve:

$$1. \quad \frac{4}{2n^2 - n - 1} + \frac{1}{2n^2 + 3n + 1} = \frac{1}{n^2 - 1}.$$

$$2. \quad \frac{1}{A - 2} + \frac{1}{A + 2} = \frac{2}{A - 3}.$$

$$3. \quad \frac{7w + 2}{6w^2 + 13w - 5} - \frac{w + 2}{2w^2 - w - 15} = \frac{2}{3w - 1}.$$

$$4. \quad \frac{6r^2 - 1}{r^2 - 9r + 14} = \frac{3r - 5}{r - 7} + 3.$$

$$5. \quad \frac{18t - 27}{14} + \frac{11t - 1}{3t + 1} = \frac{9t + 11}{7}.$$

SUGGESTION.—In equations like this, in which some denominators are monomials and others polynomials, it often is best to clear of monomial denominators only at first, and then simplify before clearing of the remaining denominators. Thus, multiplying both members by 14,

$$18t - 27 + \frac{154t - 14}{3t + 1} = 18t + 22.$$

This becomes

$$\frac{154t - 14}{3t + 1} = 49.$$

$$6. \frac{4P+3}{9} + \frac{7P-29}{5P-12} = \frac{8P+19}{18}.$$

$$7. \frac{5k+5}{k-5} = \frac{11k-15}{10} - \frac{33k+15}{30}.$$

$$8. 4 + \frac{N+7}{2(3N-4)} + \frac{9N+3}{15} = \frac{3N+6}{5}.$$

$$9. \frac{x-7}{x-9} - \frac{x-9}{x-11} = \frac{x-13}{x-15} - \frac{x-15}{x-17}.$$

SUGGESTION.—In equations of this kind much multiplication may be avoided by combining some of the fractions before clearing the equation of fractions. Thus, performing the subtractions indicated in each member,

$$\frac{-4}{x^2-20x+99} = \frac{-4}{x^2-32x+255}.$$

$$10. \frac{y-5}{y-7} + \frac{y-7}{y-9} = \frac{y-4}{y-6} + \frac{y-8}{y-10}.$$

SUGGESTION.—First transpose in order to form differences such that the fractions when combined will give simple numerators. Thus,

$$\frac{y-5}{y-7} - \frac{y-4}{y-6} = \frac{y-8}{y-10} - \frac{y-7}{y-9}.$$

$$11. \frac{B}{B-2} - \frac{B+1}{B-1} = \frac{B-8}{B-6} - \frac{B-9}{B-7}.$$

$$12. \frac{1}{v-3} - \frac{1}{v-1} = \frac{1}{v-4} - \frac{1}{v-2}.$$

$$13. \frac{n-3}{n-4} - \frac{n-4}{n-5} = \frac{n-1}{n-2} + \frac{n-2}{3-n}.$$

$$14. \frac{a+1}{a+2} + \frac{a+6}{a+7} = \frac{a+5}{a+6} + \frac{a+2}{a+3}.$$

$$15. \frac{P-1}{P-2} + \frac{P-7}{P-8} - \frac{P-5}{P-6} = \frac{P-3}{P-4}.$$

$$16. \frac{k-9}{k+5} + \frac{k-5}{k+9} = \frac{k-10}{k+4} + \frac{k-4}{k+10}.$$

$$17. \frac{2a+5}{2a+6} + \frac{2a+2}{2a+3} = \frac{2a+1}{2a+2} + \frac{2a+6}{2a+7}.$$

$$18. \frac{r-1}{r-2} = \frac{1}{r^2-r-2} + \frac{r-2}{r+1}.$$

$$19. \frac{4y+1}{4y-1} - \frac{4y-1}{4y+1} = \frac{8}{16y^2-1}.$$

$$20. \frac{z}{3-z} - \frac{3-2z}{3+z} = \frac{3(2z-7)}{6(3+z)}.$$

$$21. \frac{s-1}{s-2} = \frac{s}{s+2} + \frac{3}{s+1}.$$

$$22. \text{Solve } \frac{ax+b}{bx+c} - \frac{ax-c}{bx+2c} = \frac{b^2+c(a+b)}{b(bx+3c)} \text{ for } x.$$

$$23. \text{Solve } \frac{t^2-at}{t^2+ct-at-ac} + \frac{a-t}{t-c} = 0 \text{ for } t.$$

$$24. \text{Solve } \frac{1}{a(b-n)} + \frac{1}{b(c-n)} = \frac{1}{a(c-n)} \text{ for } n.$$

$$25. \text{Solve } \frac{xn}{2} - xy\left(\frac{x}{2} - \frac{y}{5}\right) = \frac{3y}{5}\left(n - \frac{2xy}{3}\right) \text{ for } n.$$

26. Solve $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}$ for R .

27. Solve $E = \frac{mcT + m'c't}{mc + m'c'}$ for t .

28. Solve $\frac{\frac{V}{m-n} - m}{\frac{V}{m-n} + m} + 1 = \frac{2}{1 + mn + m^2}$ for V .

29. In a guessing game, the leader says: "If you will add 20 years to your age, divide the sum by your age, add 3 to the quotient, and tell me the result, I will tell you your age." How did he find it?

SUGGESTION.—Let x = the age and a = the result given. Write an equation and solve it for x .

30. Make up a guessing game similar to that in Problem 29, and show how the age is found.

31. If a body whose weight is W pounds, moving with a velocity of V feet per second, strike a second body whose weight is w pounds and which is at rest, so that the two bodies move on together, the velocity v with which they continue together is found from the formula

$$v = \frac{W \times V}{W + w}.$$

Solve for W ; for V ; for w .

32. If a freight car, weighing 40,000 lb. and moving 8 ft. per second, strike a second car that weighs 32,000 lb. and is standing still, and the two are then coupled together, with what speed will they continue?

CHAPTER XI

PROPORTION. VARIABLES

114. Ratio.—The quotient of one number divided by another of the same kind is sometimes called their **ratio**. It usually is written in the form of a fraction, and is subject to all of the rules that apply to fractions.

Thus, the ratio of \$6 to \$100 is written $\frac{\$6}{\$100}$. The ratio of 2 lb. to 5 lb. is written $\frac{2 \text{ lb.}}{5 \text{ lb.}}$. And, in general, the ratio of a to b is written $\frac{a}{b}$.

NOTE.—An old form of writing the ratio of a to b is $a : b$. This notation is less convenient for computation than the fractional form, and is used less now than formerly.

The dividend or numerator of a ratio is sometimes called the **antecedent**, and the divisor or denominator is called the **consequent**.

EXERCISES

1. Express the ratio of 3 qt. to 6 qt. as a fraction in its lowest terms.

2. If the rate of interest on a sum of money is 5 %, that is, the ratio of \$5 to \$100, express the rate as a fraction in its lowest terms.

3. The death rate in Boston in a recent year was 18 to 1000 population. Express this ratio as a fraction.

4. An alloy consists of copper and tin in the ratio of 2 to 3. What part of it is each?

5. A solution consists of alcohol and water in the ratio of 3 to 5. What part of it is water?

6. In a city of 84,000 population the number of births in a year is 560. What is the birth rate? How many per 1000?

7. When a man 5 ft. 10 in. tall casts a shadow 35 ft. long, find the ratio of his height to the length of his shadow.

8. The *specific gravity* of a solid or liquid is the ratio of the weight of any volume of it to the weight of an equal volume of water. A cubic foot of water weighs 62.5 lb. A cubic foot of steel weighs 490 lb. Find the specific gravity of steel. Express the answer first as a common fraction, then as a decimal computed to tenths.

9. Find the specific gravity, computed decimally to tenths, of each of the following substances:

SUBSTANCE	WEIGHT IN POUNDS OF 1 CU. FT.	SUBSTANCE	WEIGHT IN POUNDS OF 1 CU. FT.
Cast Iron	450.0	Oak	65.0
Brass	523.8	Ash	52.8
Gold	1200.9	Cork	15.0

10. The diameter of a circular plate is 6 in., and the circumference is measured with a tape line and found to be 18.85 in. Find the ratio of the circumference to the diameter.

11. A boy measures the wheel of his bicycle, and finds that its diameter is 28 in. and circumference 88 in. What is the ratio of the circumference to the diameter?

12. Simplify the ratio of $x + y$ to $x^2 - y^2$ by writing it as a fraction reduced to its lowest terms.

13. Simplify the ratio of $\frac{a}{b}$ to $\frac{b'}{a'}$ by writing it as a simple fraction.

14. Which ratio is the greater, $\frac{7}{12}$ or $\frac{26}{45}$?

SUGGESTION. — Reduce them to a common denominator, and compare the numerators.

15. Write in descending order of magnitude $\frac{5}{6}$, $\frac{13}{15}$, $\frac{21}{25}$.

16. Two partners in business, A and B, divide a profit of \$ 2135 between them so that A's part and B's part are in the ratio of 3 to 4. How many dollars does each receive?

SUGGESTION. — Let d be the amount that A receives.

Then, $2135 - d$ is the amount that B receives.

Hence,
$$\frac{d}{2135 - d} = \frac{3}{4}.$$

17. If two partners, A and B, divide a profit of \$1200 in the ratio of 1 to 2, how many dollars does each receive?

18. In a certain city with a population of 247,520, the ratio of the Germans to all other nationalities together is 2 to 15. What is the German population?

19. Separate 40 into two parts which are in the ratio of 3 to 7.

20. Separate 85 into two parts which are in the ratio of 5 to 12.

21. The sides of a triangle are 5 in., 9 in., and 7 in. Divide the side 7 in. long into two parts whose ratio equals the ratio of the other two sides.

115. Proportion. — An equation whose members consist of two ratios is called a **proportion**.

Thus, $\frac{3}{4} = \frac{6}{8}$, $\frac{5a}{6b} = \frac{15a}{18b}$, and $\frac{a}{b} = \frac{c}{d}$ are proportions.

The proportion $\frac{a}{b} = \frac{c}{d}$, is read either " a over b equals c over d ," or " a is to b as c is to d ."

The numbers forming one of the ratios are said to be "proportional to" the numbers forming the other ratio; or the four numbers forming the ratios are said to be "in proportion."

In any proportion $\frac{a}{b} = \frac{c}{d}$, a , b , c , and d are called the **terms**. The first and fourth terms, a and d , are called the **extremes**; and the second and third terms, b and c , are called the **means**.

Since a proportion is an equation, all of the operations which

may be performed upon an equation, such as clearing of fractions, etc., may be performed upon a proportion.

NOTE. — Since the ratio of any two concrete numbers is the same as that of the corresponding abstract numbers, in performing any operation upon a proportion, confusion may be avoided by using only abstract values for the terms.

EXERCISES

Find the value of the unknown term in each of the following proportions:

$$1. \frac{n}{2} = \frac{15}{10}.$$

$$4. \frac{7}{9} = \frac{42}{t}.$$

$$7. \frac{36}{v} = \frac{1}{4}.$$

$$2. \frac{x}{12} = \frac{1}{8}.$$

$$5. \frac{3}{10} = \frac{A}{40}.$$

$$8. \frac{9}{16} = \frac{12}{P}.$$

$$3. \frac{4}{y} = \frac{16}{21}.$$

$$6. \frac{24}{35} = \frac{r}{70}.$$

$$9. \frac{20}{36} = \frac{8}{L}.$$

Find the values of n in the proportions:

$$10. \frac{a}{b} = \frac{c}{n}.$$

$$12. \frac{t}{v} = \frac{n}{w}.$$

$$14. \frac{1}{r} = \frac{r}{n}.$$

$$11. \frac{x}{n} = \frac{y}{z}.$$

$$13. \frac{n-a}{a} = \frac{n+b}{b}.$$

$$15. \frac{an}{b} = \frac{b}{c}.$$

16. A family of three members and a family of four members camp out together. The total cost of provisions is \$112, which they wish to divide in proportion to the sizes of the families. How much must each family pay?

116. Definitions. — In any proportion $\frac{a}{b} = \frac{c}{d}$, the fourth term d is called a **fourth proportional** to the other three terms, a , b , and c .

A proportion such as $\frac{a}{x} = \frac{x}{b}$, in which the means are equal, is called a **mean proportion**. The mean term x is called the **mean proportional** between the extremes a and b .

In a mean proportion such as $\frac{a}{x} = \frac{x}{b}$, b is called the **third proportional** to a and x .

EXERCISES

Find the fourth proportional to:

1. 2, 3, and 10.

SUGGESTION. — Let x be the fourth proportional. Then $\frac{2}{3} = \frac{10}{x}$.

2. 4, 5, and 12.

5. 1, a , and a^2 .

3. 7, 10, and 28.

6. m , n , and m^2n .

4. 16, 12, and 4.

7. 1, $1 - t$, and $1 + t$.

Find the third proportional to:

8. 3 and 6.

SUGGESTION. — Let x be the third proportional. Then $\frac{3}{6} = \frac{6}{x}$.

9. 8 and 12.

12. a and b .

10. x and $3x^3$.

13. $1 + n$ and $1 - n^2$.

11. P and Pt .

14. 10 and 1.

Find the mean proportional between:

15. 2 and 8.

SOLUTION. — Let x be the mean proportional. Then $\frac{2}{x} = \frac{x}{8}$.

Clearing of fractions, $x^2 = 16$.

Taking square root, $x = \pm 4$.

16. 3 and 12.

19. n and n^5 .

17. 7 and 28.

20. $2A^3$ and $50A$.

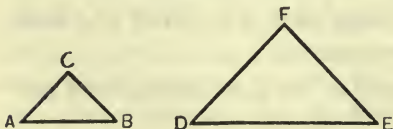
18. 9 and 1.

21. R and $R + 2R^2 + R^3$.

117. Distances found by Proportion. — Two triangles that are drawn with the angles of one equal to the corresponding angles of

the other have the same shape, and are called **similar triangles**, as ABC and DEF .

Let the student draw two such triangles, making the



side DE of one three times as long as the corresponding side AB of the other. Measure the other sides, and compare them. It will be found that:

In two similar triangles, the corresponding sides are proportional.

Thus, in the above figure, $\frac{AB}{DE} = \frac{AC}{DF}$, etc.

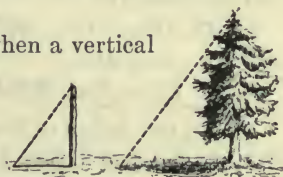
This principle is much used in computing unknown distances.

EXERCISES

1. In two similar triangles, the sides of the larger are 12 in., 14 in., and 20 in., and the shortest side of the smaller is 3 in. Find the other sides of the smaller triangle.

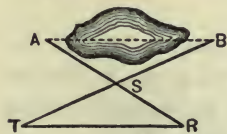
2. The sides of a triangular field are 16 rd., 24 rd., and 30 rd. The smallest side of a similar field is 40 rd. Find the other two sides.

3. A tree casts a shadow 48 ft. long when a vertical rod 6 ft. high casts a shadow 4 ft. long. How high is the tree?



4. A church spire casts a shadow 23 ft. long when a man 5 ft. 10 in. tall who is passing by casts a shadow 2 ft. long. Find the height of the spire.

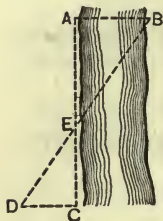
5. The distance from A to B on opposite sides of a lake may be found as follows:



The distances from A to R and from B to T are measured off, making the triangles ASB and TSR similar. If AS is taken 400 yd., and SR 300 yd., and TR is measured and found to be 580 yd., how far is it from A to B ?

6. If A and B are two points on opposite sides of a hill, and out in the plain at the foot of the hill distances are measured as in Problem 5, so that AS is taken 4050 ft. and SR 1350 ft., and TR is found to be 1600 ft., how far is it between A and B ?

7. To find the distance AB across a stream, measure off a distance AC several yards long, along the bank. Then at C measure off a distance CD at right angles to AC . By sighting across from D to B , locate a stake at a point E of AC in line with D and B . Get the lengths of AE and EC . Since the triangles are similar, write the proportion by means of which AB may be computed.



If AE is 80 yd., EC 20 yd., and DC 15 yd., what is the width of the stream?

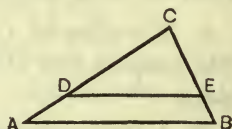
8. It is known in geometry that a line parallel to one side of a triangle divides the other two sides into four proportional parts. That is, if DE is parallel to AB , $\frac{CD}{DA} = \frac{CE}{EB}$.

(1) If $AD = 6$ in., $DC = 3$ in., and $BE = 4$ in., find EC .

(2) If $AD = 10$ in., $BE = 12$ in., and $EC = 20$ in., find DC .

(3) If $DC = 9$ in., $BE = 7$ in., and $EC = 5$ in., find AD .

9. In the figure of Problem 8, if $AD = 8$ in., $DC = 5$ in., and $BC = 26$ in., find BE and EC .

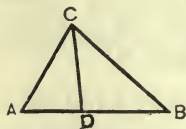


SUGGESTION. — Let $BE = x$. Then $EC = 26 - x$.

Hence, $\frac{8}{5} = \frac{x}{26 - x}$.

10. In the figure of Problem 8, if $AC = 40$ ft., $BE = 12$ ft., and $EC = 15$ ft., find AD and DC .

11. It is known in geometry that in any triangle ABC , if the line CD divides the angle at C into two equal parts, it divides the opposite side into parts proportional to the other two sides; that is, $\frac{AD}{DB} = \frac{AC}{BC}$.

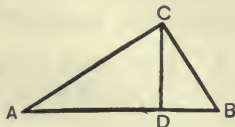


If $AC = 15$ in., $BC = 10$ in., and $AB = 12$ in., find AD and DB .

12. In the figure of Problem 11, if $AC = 40$ yd., $BC = 32$ yd., and $AB = 60$ yd., find AD and DB .

13. In the triangle ABC , the angle at C is a right angle, and CD is perpendicular to AB . It is known in geometry that CD is a mean proportional between AD and DB .

If $AD=20$ in., and $DB=5$ in., find CD .



14. A method used several hundred years ago of finding the distance from A to the inaccessible point B was to erect a vertical staff AC , place upon it an instrument resembling a carpenter's square, pointing one blade towards B , and note the place D on the ground to which the other blade pointed. AC and DA were measured. If $AC=52$ in., and $DA=6$ in., find AB .

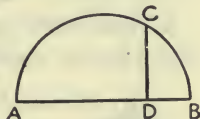


15. In the semicircle with the diameter AB , CD , a perpendicular to AB , is a mean proportional between AD and DB .

If $AD=12$ in., and $DB=3$ in., find CD .

If $AD=8$ in., and $CD=6$ in., find DB .

If $AB=20$ in., and $CD=8$ in., find AD and DB .



118. **Proportion in Simple Machines.** — Simple machines, such as the lever, the wheel and axle, etc., are instruments by means of which a small **effort** exerted may be made to overcome a **resistance** of much greater size. Proportion is involved in the use of simple machines.

EXERCISES

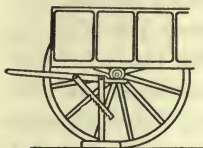
1. A lever is a stiff bar of wood or metal that is movable about a fixed point or pivot called the fulcrum. The resistance W to be overcome is applied at one end of the bar, and the effort w , exerted to overcome it, at the other end. If D is the distance of W from the fulcrum, and d the distance of w from the fulcrum, then

$$\frac{w}{W} = \frac{D}{d}.$$

What effort is required to lift a weight of 600 lb. by a lever, if the weight is 8 in. from the fulcrum and the effort applied 40 in. from the fulcrum?

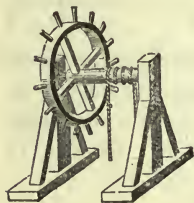
2. What effort is required to lift a stone weighing 840 lb. by means of a crowbar 60 in. long, if the fulcrum is placed 4 in. from the stone?

3. A blacksmith weighing 180 lb. lifts wagons, etc., by means of a wagon jack. The weight lifted is applied at a point 6 in. from the pivot or fulcrum, and he grasps the handle 32 in. from the pivot. By throwing his whole weight upon the jack, how much can he lift?



4. The entire length of a lever is 48 in. Where must the fulcrum be placed in order that a resistance of 360 lb. may be overcome by an effort of 120 lb.?

5. The wheel and axle consists of a wheel and a cylindrical axle passing through its center, the two being fastened rigidly together, so that the axle is turned by revolving the wheel. The weight or resistance to be overcome is applied at the circumference of the axle through a cord wrapped around the axle, and the effort required to overcome it is applied at the circumference of the wheel. If the resistance is W , the effort required to overcome it w , the radius of the axle R , and the radius of the



wheel r , then $\frac{w}{W} = \frac{R}{r}$.

If the radius of the wheel is 20 in., the radius of the axle 6 in., and the resistance to be overcome 500 lb., what is the effort required?

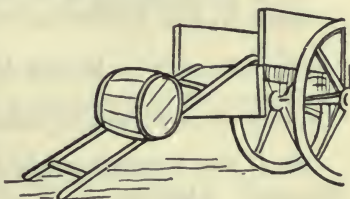
6. The axle of a windlass shown in the above figure, which is used for drawing water from a well, has a radius of $4\frac{1}{2}$ in., and the radius of the wheel 15 in. What effort is required to lift a bucket of water weighing 54 lb.?

7. A capstan, a form of wheel and axle used in lifting anchors on ships, has an axle 12 in. in diameter and a lever arm or spoke 45 in. long. How much effort is required to lift by it an anchor weighing a ton?

8. The inclined plane is a smooth sloping surface that is used in raising a heavy object through the application of a comparatively small effort, the object being raised by sliding or rolling it along the surface of the plane. If the effort is applied parallel to the surface of the plane, the effort applied is to the weight lifted as the height of the plane is to its length; that is, $\frac{w}{W} = \frac{H}{L}$, where

w = the effort applied, W = the weight, H = the height of the plane, or distance the weight is lifted, and L = the length of the plane.

If the length of an inclined plane is 12 ft., the height 3 ft., and the weight of the object lifted 400 lb., what effort is required to move it up the plane?



9. A barrel of material weighing 180 lb. is loaded into a wagon bed 32 in. from the ground by rolling it up a board 10 ft. long with one end resting on the ground and the other on the wagon bed. What effort is required?

10. Two men place a 580 lb. building stone upon a wagon by sliding it up a board 8 ft. long, one end of the board being on the ground and the other resting on the bed of the wagon. The wagon bed is 20 in. from the ground. What effort do they use in loading the stone in addition to that required to overcome the friction?

11. Ice is stored in an ice house by dragging it up an inclined plane. The incline is 200 ft. long and 40 ft. high. What effort is required to pull up a block of ice weighing 250 lb.?

12. A locomotive pulls a train weighing 720 tons up a grade with a rise of $2\frac{1}{2}$ ft. to 100 ft. of grade. How much more pull must the locomotive exert than if the train were on a level road bed?

119. Important Principles in Proportion.—The following important principles in proportion are often used in geometry, and elsewhere, where proportion is applied. Let the pupil establish each of them.

(1) *If four numbers are in proportion, the product of the extremes equals the product of the means.* That is,

$$\text{if } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc.$$

SUGGESTION. — Clear $\frac{a}{b} = \frac{c}{d}$ of fractions.

(2) *If the product of two numbers equals the product of two other numbers, the four numbers are in proportion.* That is,

$$\text{if } ad = bc, \text{ then } \frac{a}{b} = \frac{c}{d}.$$

SUGGESTION. — Divide both members of $ad = bc$ by bd .

(3) *If four numbers are in proportion, they are in proportion by inversion.* That is,

$$\text{if } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{b}{a} = \frac{d}{c}.$$

SUGGESTION. — Divide $1 = 1$ by the members of $\frac{a}{b} = \frac{c}{d}$.

(4) *In any proportion, the means may be interchanged, or the extremes interchanged, without destroying the proportion.* That is,

$$\text{if } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a}{c} = \frac{b}{d} \text{ and } \frac{d}{b} = \frac{c}{a}.$$

SUGGESTION. — Multiply both terms of $\frac{a}{b} = \frac{c}{d}$ by $\frac{b}{c}$; by $\frac{d}{a}$.

(5) *The terms of any proportion are in proportion by addition.* That is,

$$\text{if } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a+b}{b} = \frac{c+d}{d}.$$

SUGGESTION. — Add 1 to each member of $\frac{a}{b} = \frac{c}{d}$.

(6) *The terms of any proportion are in proportion by subtraction.*
That is,

$$\text{if } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a-b}{b} = \frac{c-d}{d}.$$

SUGGESTION. — Subtract 1 from each member of $\frac{a}{b} = \frac{c}{d}$.

(7) *Like powers of the terms of a proportion are in proportion.*
That is,

$$\text{if } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a^2}{b^2} = \frac{c^2}{d^2}, \frac{a^3}{b^3} = \frac{c^3}{d^3}, \text{ etc.}$$

SUGGESTION. — Raise both members of $\frac{a}{b} = \frac{c}{d}$ to the same power.

(8) *If two or more ratios are equal, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*
That is,

$$\text{if } \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \text{etc.}, \text{ then } \frac{a+c+e+\text{etc.}}{b+d+f+\text{etc.}} = \frac{a}{b} = \text{etc.}$$

For, since $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \text{etc.}$, let each ratio equal r .

Then $a = rb, c = rd, e = rf, \text{ etc.}$

Hence, $a + c + e + \text{etc.} = rb + rd + rf + \text{etc.}$
 $= r(b + d + f + \text{etc.}).$

Hence, $\frac{a+c+e+\text{etc.}}{b+d+f+\text{etc.}} = r = \frac{a}{b} = \frac{c}{d} = \text{etc.}$

EXERCISES

1. If $ad = bc$, show that $\frac{a}{c} = \frac{b}{d}$.

2. If $ad = bc$, show that $\frac{d}{b} = \frac{c}{a}$.

3. If $ad = bc$, show that $\frac{d}{c} = \frac{b}{a}$.

4. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

SUGGESTION.—Divide the members of $\frac{a+b}{b} = \frac{c+d}{d}$ (5) by the members of $\frac{a-b}{b} = \frac{c-d}{d}$ (6).

5. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{c+d} = \frac{b}{d}$.

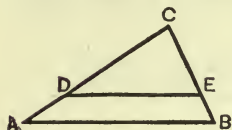
6. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-b}{c-d} = \frac{b}{d}$.

7. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{3a+b}{b} = \frac{3c+d}{d}$.

8. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-2b}{b} = \frac{c-2d}{d}$.

9. It is known that, in triangle ABC , if DE is parallel to AB ,

$$\frac{AD}{DC} = \frac{BE}{EC}. \quad \text{Show from this that } \frac{AC}{DC} = \frac{BC}{EC}.$$



10. In the similar triangles in § 117, it is

$$\text{known that } \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}.$$

Show from this that the ratio of the perimeters of the triangles equals the ratio of any two corresponding sides, such as AB and DE . See § 119, (8).

120. Variable Quantities. — Any quantity such as one's age, the temperature, the population of the country, etc., that is always changing is called a **variable quantity**, or a **variable**. Any quantity that does not change is called a **constant**.

Many of the quantities encountered in business, in science, etc., are variables.

Thus, the prices at which many foodstuffs are retailed usually vary from day to day, or from week to week. Butter may sell at 24¢ one week, 27¢ the next week, 29¢ the next week, etc.

As an illustration of variable quantities encountered in science, suppose that a small heavy object such as a lead ball be dropped from a great height toward the earth. The farther it falls, the faster it falls. At the end of one second its speed will be about 32 ft. per second, at the end of the second

second its speed will be 64 ft. per second, at the end of the third second its speed will be 96 ft. per second, etc. That is, the speed is a variable constantly increasing.

121. Related Variables.—Some variable quantities are so related to each other that the values of one depend upon the values of the other.

Thus, if a train runs 40 miles an hour, the distance that it has traveled depends upon the time elapsed since it started. As the time increases the distance increases.

The relation between two such variables may be expressed by an equation.

Thus, in the above example, if the distance in miles is denoted by d , and the time in hours by t , then the relation between them is expressed by $d = 40t$.

One of the simplest cases of related variables is that in which the *ratio of two variables is constant*. If x and y are the variables, then $\frac{x}{y} = k$, or $x = ky$, where k is a constant. In this case it is said that " x varies directly as y ."

It is evident that *when one quantity varies directly as another, the ratio of any two corresponding values of the variables equals the ratio of any other two corresponding values*, since each ratio equals the same constant.

Thus, if x varies directly as y , and x_1 and y_1 , and x_2 and y_2 are any two pairs of corresponding values of x and y , then

$$\frac{x_1}{y_1} = \frac{x_2}{y_2}.$$

Hence, the fact that x varies directly as y is sometimes expressed by saying that " x is proportional to y ."

EXAMPLE.— m varies directly as n , and when m is 3, n is 5. Write the equation expressing the relation between m and n . Find m when n is 15.

$$\frac{m}{n} = \frac{3}{5}.$$

If n is 15,

$$\frac{m}{15} = \frac{3}{5}.$$

Solving,

$$m = 9.$$

EXERCISES

1. If a train runs 35 miles an hour, write the equation expressing the relation between the variable time t and the distance d .

2. If the price of coffee is 30 ¢ a pound, the cost of any quantity varies as the number of pounds bought. If c = cost and w = weight in pounds, express the relation between them by an equation.

3. If the price of sugar is 5 ¢ a pound, express by an equation the relation between the number of pounds bought and the cost.

4. If the rate of interest is 6 %, the amount of interest on any sum of money varies as the time. Express by an equation the relation between the interest and the time.

5. The velocity of a body let fall toward the ground varies as the time during which it has fallen from rest, and the velocity at the end of 3 sec. is 96 ft. per second. As the time increases, the velocity increases at the same rate. Write the equation between the velocity and time.

6. The force with which a moving body of any given velocity strikes a stationary body varies as the mass or weight of the moving body. If I strike a nail with a force of 15 lb. by using a hammer weighing $\frac{1}{2}$ lb., with what force would a 2 lb. hammer strike it when swinging with the same speed?

7. The resistance offered by a wire of given size to a current of electricity varies directly as the length of the wire. If a wire 50 ft. long gives a resistance of 6 ohms, what will be the resistance of 75 ft.?

8. If R is the resistance in Problem 7 and L the length of wire, write the equation between them.

9. If x varies directly as y , and $x = 2$ when $y = 9$, find x when $y = 36$. Write the equation between x and y .

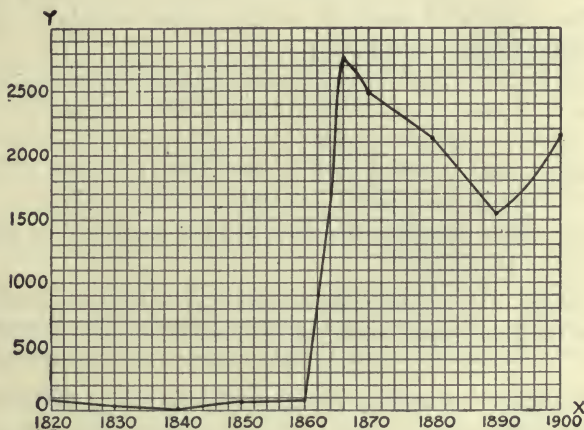
10. If P varies directly as V , and $P = 7$ when $V = 5$, write the equation between P and V . Find V when $P = 35$.

122. Variation shown by Graphs.—In dealing with variable quantities, their variation often is depicted to the eye by means of a **chart** or **graph**. See § 29.

For example, the public debt of the United States, in millions of dollars, from 1820 to 1900 is given in the following table :

YEAR	1820	1830	1840	1850	1860	1866	1870	1880	1890	1900
Debt	91	49	4	63	65	2773	2480	2120	1552	2136

On squared paper draw a horizontal line OX and a line OY at right angles to it. These are called the **axes**. The point O is called the **origin**.



On the line OX let a distance of one space represent 2 years of time. Then the dates in the above table will be represented on OX as in the chart. On OY let a distance of one space represent 100 million dollars. On the 1820 vertical line measure off a distance representing 91 million dollars, and mark a point. On the 1830 vertical line measure off a distance representing 49 million dollars, and mark a point. Similarly, locate a point on each of the other date lines corresponding to the amount of the debt of that date. Draw a line connecting all of the points obtained.

This curved line, called the *graph* of the national debt, depicts clearly to

the eye the variation of the national debt throughout the period of time considered.

NOTE. — Various kinds of apparatus are made for constructing graphs automatically. The *thermograph* or recording thermometer is a good illustration. A pen connected to a thermometer moves up and down as the temperature rises and falls. At the same time a clock mechanism runs a strip of ruled paper under the pen, so that the pen traces a continuous curve or graph on the paper.

Among other instruments for making graphs is the *seismograph*, used for recording earthquakes.

The pupil should be provided with several sheets of squared paper for use in the exercises of this and subsequent chapters.

EXERCISES

1. The number of teachers in the public schools of Chicago since 1850 has been as follows:

YEAR	1850	1860	1870	1880	1890	1900	1909
Teachers	21	123	557	898	2711	5806	6296

Draw a graph showing the growth in the size of the teaching force during that time.

SUGGESTION. — On squared paper draw axes as in the example above. On one represent the time, letting one space represent 2 years. On the other represent the number of teachers, letting one space represent 200 teachers.

2. The following average heights of children at different ages have been determined from measurements of thousands of individuals:

AGE	$\frac{1}{2}$ Yr.	1 Yr.	2 Yr.	3 Yr.	4 Yr.	5 Yr.	6 Yr.	7 Yr.	8 Yr.
Height	24 in.	29 in.	36 in.	40 in.	42 in.	44 in.	46 in.	48 in.	50 in.

Draw a graph showing the rate of growth of the average individual.

3. The population, to the nearest million, of the United States since 1800 is given by decades as follows:

1800, 5	1840, 17	1880, 50
1810, 7	1850, 23	1890, 63
1820, 10	1860, 31	1900, 76
1830, 13	1870, 39	1910, 93

Draw a graph showing the growth of the population.

4. The population, to the nearest thousand, of the city of Chicago since 1840 is recorded as follows:

1840, 4	1860, 109	1880, 503	1900, 1699
1850, 28	1870, 299	1890, 1100	1910, 2185

Draw a graph showing the growth of the city.

5. The following table gives, in billions of dollars, the total value of the farm property and products of the United States by decades from 1850 to 1900:

YEAR	1850	1860	1870	1880	1890	1900
Value	4	8	11	12	16	21

Show by a graph the growth of agriculture in the United States.

6. The amount of the annual exports of the United States, in millions of dollars, has been as follows:

1790, 20	1820, 70	1850, 152	1880, 853
1800, 71	1830, 74	1860, 400	1890, 910
1810, 67	1840, 132	1870, 451	1900, 1499

Show the growth of our exports by a graph.

7. The temperature at a place was recorded as follows: 6 A.M., 8°; 7 A.M., 8°; 8 A.M., 10°; 9 A.M., 11°; 10 A.M., 14°; 11 A.M., 18°; 12 M., 20°; 1 P.M., 20°; 2 P.M., 20°; 3 P.M., 18°; 4 P.M., 16°; 5 P.M., 15°; 6 P.M., 13°. Draw a graph to show the variation of temperature during the 12 hours.

123. Price Curves, etc. — Graphs may be constructed and used for determining costs of different quantities of goods, interest on money for different periods of time, etc., without computation. This is shown by an example.

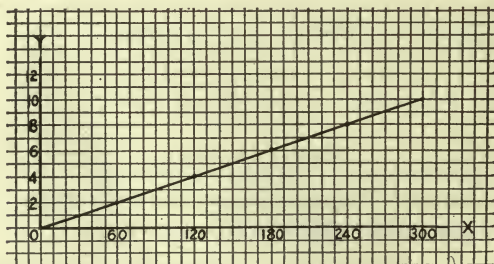
If eggs sell at 30 ¢ a dozen, the relation between the number of dozens and the cost may be expressed by the equation

$$c = 30d,$$

where d is the number of dozens and c is their cost. If values are given to d , corresponding values may be found for c , as given in the table:

d	0	2	4	6	8	10	12
c	0	60	120	180	240	300	360

On squared paper draw two axes, OX and OY , at right angles, as in § 122.



On OY let a space represent 1 doz., and on OX let a space represent 10 ¢. Then, on the 60 ¢ line mark a point representing 2 doz. On the 120 ¢ line mark a point representing 4 doz., etc. Draw a line through the points thus marked. It is seen that this line or graph is a straight line. It is called the "price curve." Most price curves are straight lines.

By looking at this price curve we can get the cost of any number of dozens, even of a fractional number. For example, to find the cost of 7 doz., observe the point where the horizontal line 7 spaces up meets the price curve; observe the point directly beneath this on the axis OX ; this is 21 spaces from O , and hence represents \$2.10. Similarly, the cost of $9\frac{1}{2}$ doz. is seen to be \$2.85.

EXERCISES

1. Since the price curve in the example in § 123 is a straight line, how many of the points would have to be located through which to draw it? Should these be taken close together or far apart, in order to get the position of the price curve most exact?

2. From the price curve in § 123, give the costs of the following: 3 doz.; 5 doz.; $10\frac{1}{2}$ doz.; $6\frac{3}{4}$ doz.; $8\frac{1}{2}$ doz.

3. If eggs sell at 25¢ a dozen, draw the price curve.

4. On this price curve, find the cost of 11 doz.; of $7\frac{1}{2}$ doz.; of $9\frac{3}{4}$ doz.

5. If coffee sells at 28¢ a pound, draw the price curve.

6. On this price curve find the cost of $8\frac{1}{2}$ lb.; 17 lb.; 24 lb.

7. If sugar sells at $5\frac{3}{4}$ ¢ a pound, draw the price curve.

8. On this price curve find the cost of 16 lb.; of 42 lb.; of $8\frac{1}{2}$ lb.

9. If money is loaned at 4% interest, draw a graph showing the interest on \$1 for different periods of time.

SUGGESTION. — Let distances on one axis represent time, a space denoting a month; and let distances on the other axis represent interest on \$1, a space denoting 1¢.

10. On this graph find the interest on \$1 for 18 mo.; for 9 mo.; for 22 mo.; for 3 yr. 5 mo.; for 5 yr. 2 mo.

11. On the graph in Problem 9 find the interest on \$450 at 4% for 15 mo.

SUGGESTION. — Find the interest on \$1, and multiply this by 450.

12. On the graph in Problem 9 find the interest on \$275 at 4% for 1 yr. 7 mo.; on \$1260 at 4% for 3 yr. 10 mo.

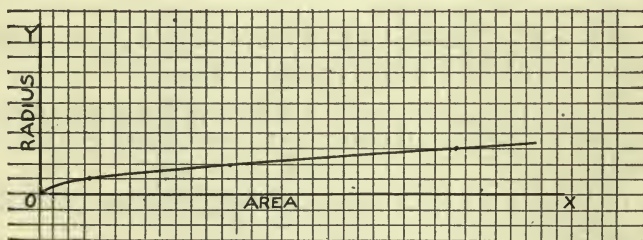
13. If money is loaned at 6%, draw a graph showing the interest on \$1 for different periods of time. Use this to find the interest on different sums of money at 6% for different lengths of time.

14. The relation between the circumference of a circle and the diameter is expressed by $c = \pi d$, where c denotes the circumference and d the diameter. Using $\pi = 3\frac{1}{7}$, draw the graph expressing this relation. On this graph give at sight the approximate values of the circumferences of circles whose diameters are 4 in.; $5\frac{1}{4}$ in.; $8\frac{3}{4}$ in.

15. The area of a circle is found from $A = \pi R^2$, where A denotes the area and R the radius. Using $\pi = 3\frac{1}{7}$, a few values of A for corresponding values of R are given in the table:

R	0	1	2	3	4
A	0	$3\frac{1}{7}$	$12\frac{2}{7}$	$28\frac{2}{7}$	$50\frac{2}{7}$

Representing values of the radius R on the axis OY , one space denoting 1 in., and values of the area A on the axis OX , one space denoting 1 sq. in., the graph showing the relation between the



radius and the area is found to be the *curve* in the figure. The approximate value of the area of a circle with any given radius or of the radius of a circle with any given area, may be determined at sight by this graph.

Give at sight the area of a circle whose radius is $1\frac{1}{2}$ in.; $2\frac{1}{2}$ in.

Give at sight the radius of a circle whose area is 10 sq. in.

NOTE. — Observe that the graph in Problem 16 is not a straight line. Many problems lead to graphs that are not straight lines. Some of these will be encountered in more advanced work.

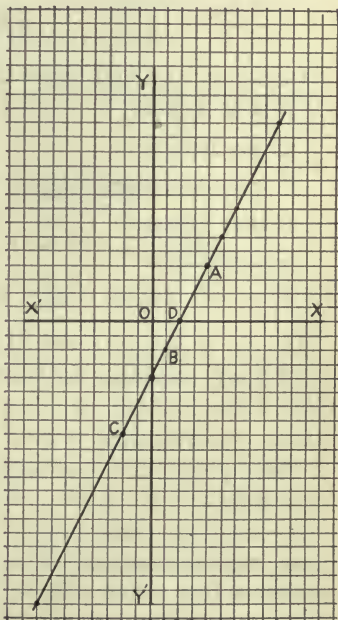
124. Graphs of Linear Equations.—As shown in the preceding sections, the relation between two quantities may be expressed by an *equation*, also the relation may be shown by a *graph*. The graph is often spoken of as **the graph of the equation**.

The two literal quantities in an equation may, in some cases, represent *negative* as well as *positive* values. The process of drawing the graph of an equation between two quantities which have both positive and negative values will be given in this section. The construction of such a graph is given below.

x	2	4	5	7	10	1	0	-2	-5	-7
y	0	4	6	10	16	-2	-4	-8	-14	-18

Let the equation be $2x = y + 4$. By assigning values to y , compute the corresponding values of x and tabulate the results as shown in the table. Thus, when $y = 0$, the equation becomes $2x = 4$, from which $x = 2$. When $y = 6$, $2x = 10$, from which $x = 5$. When $y = -8$, $2x = -4$, from which $x = -2$, etc.

Two axes, XX' and YY' , are drawn at right angles and meeting at O . Corresponding to each set of values of x and y , a point is located, as in § 122 and § 123, the values of x being measured along or parallel to XX' , and the values of y along or parallel to YY' . Positive values of x are measured to the right of YY' and negative values to the left, as in § 28. Positive values of y are measured above XX' and negative values below XX' . For example, the point A corresponding to $x = 4$ and $y = 4$ is obtained by measuring 4 spaces to the right and 4 spaces upward. The point B corresponding to $x = 1$ and $y = -2$ is obtained by measuring 1 space to the right and 2 spaces downward. The point C corresponding to $x = -2$ and $y = -8$ is obtained by measuring 2 spaces to the



left and 8 spaces downward. The point D corresponding to $x=2$ and $y=0$ is obtained by measuring 2 spaces to the right on XX' ; etc. By drawing a line through all of the points located the graph of the equation is obtained.

It is seen that the graph of the above equation is a straight line. It is shown in more advanced mathematics, and may be assumed here, that

The graph of every linear equation containing two variables is a straight line.

NOTE. — It is seen that in locating a point corresponding to a set of values of the two variables in an equation, the signs of the numbers serve to tell the *directions* of the point from the two axes. This is analogous to the method of locating a point on the earth's surface by knowing its longitude and latitude. If the latitude of a place is minus, the place is south of the equator, etc.

EXERCISES

Draw two axes and locate the following points:

1. $x=4, y=6.$

7. $x=-6, y=5.$

2. $x=10, y=7.$

8. $x=-10, y=12.$

3. $x=15, y=2.$

9. $x=9, y=-4.$

4. $x=0, y=8.$

10. $x=15, y=-5.$

5. $x=12, y=0.$

11. $x=-6, y=-6.$

6. $x=0, y=0.$

12. $x=-10, y=-12.$

13. By assigning eight different values to y in the equation $x-2y=7$, including at least two negative values, compute the eight corresponding values of x . Draw axes and locate the eight points corresponding to these sets of values of x and y . Are all of these points in one straight line? Draw a line through all of the points.

By assigning eight values to y in each of the following equations, including at least three negative values, and computing the corresponding values of x , locate eight points of the graph of the equation, and draw the graph.

14. $x = y$.

16. $x + 4y = 10$.

18. $5x - 4y = 20$.

15. $x + y = 0$.

17. $3x - y = 12$.

19. $2x + 3y = 12$.

20. Since the graph of a linear equation in two variables is a straight line, how many points of it must be located in order to draw it accurately?

By locating only two points of each, draw the graphs of all of the following equations on one pair of axes:

21. $3x - y = 12$.

23. $2x - 4y = 5$.

25. $5x - y = 4$.

22. $y = 2x$.

24. $2x + 4y = 7$.

26. $4x = 3y$.

27. If the cost of setting the type for printing a circular is 75¢, and the cost of paper and press work in printing it is $\frac{1}{4}$ ¢ a copy, then

$$c = \frac{1}{4}n + 75,$$

where c denotes the cost in cents of printing any number of copies and n denotes the number of copies printed.

Draw the graph of this equation.

Give at sight from the graph the cost of 12 copies; 20 copies; 100 copies.

SUPPLEMENTARY EXERCISES

1. Express in simplest form the ratio of $m - \frac{n^2}{m}$ to $1 - \frac{n}{m}$.

2. Separate 72 into three parts which are in the ratio of 2:3:4.

3. If n is a positive number, which is the greater ratio, $\frac{3+3n}{3+4n}$

or $\frac{3+4n}{3+5n}$?

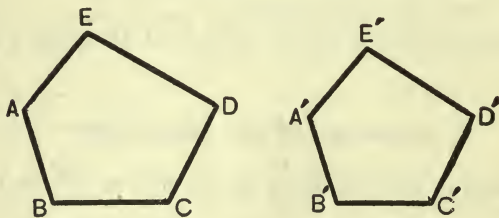
SUGGESTION. — Reduce to a common denominator, then compare the numerators.

4. If x and y are positive numbers, which is the greater ratio,

$\frac{x+5y}{x+6y}$ or $\frac{x+7y}{x+8y}$?

5. Find the mean proportional between $4a^3$ and $9a(a-b)^2$.
6. Find the fourth proportional to $v^2 - w^2$, $\frac{1}{v+w}$, and $v-w$.
7. Write in three ways the proportion between 3, 4, 20, and 15.
8. Write as a proportion $x^2 - y^2 = 5 \times 6$.
9. If $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$, show that $\frac{a+3b+5c}{a'+3b'+5c'} = \frac{a}{a'}$.
10. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a^2 - b^2}{b^2} = \frac{c^2 - d^2}{d^2}$.
11. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a}{a+c} = \frac{a+b}{a+b+c+d}$.
12. In the similar polygons $ABCDE$ and $A'B'C'D'E'$ it is known that

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}.$$



Show that the ratio of the perimeters of the polygons equals the ratio of any two corresponding sides.

13. x varies as y , and when $x=2$, $y=5$. Find y when $x=15$.
14. The distance that a body falls from rest varies as the square of the time. In 2 seconds it falls 64 ft. How far will it fall in 3 sec.? In 4 sec.? In 10 sec.?
15. One quantity is said to *vary inversely* as another when, during all of their changes, their product remains constant. As one increases, the other decreases. If x varies inversely as y , then $xy=k$, where k is a constant.

If x varies inversely as y , and $y=2$ when $x=4$, find y when $x=32$.

16. The volume of any gas varies inversely as the pressure upon it. When the pressure is 8 lb. the volume is 8 cu. in. What is the volume when the pressure is 4 lb.?

17. The number of vibrations made by the pendulum of a clock in a given time varies inversely as the square root of its length. A pendulum 39.1 inches long makes one vibration in a second. How long must a pendulum be to make 4 vibrations in a second?

CHAPTER XII

SYSTEMS OF LINEAR EQUATIONS

125. Systems of Equations.—In Chapter VI it was shown that some problems may be expressed by means of two linear equations containing two unknown numbers. These equations were called *simultaneous*, and were said to form a *system*. It was found that the equations in each system in that chapter had one set of values of the unknown numbers that satisfied both equations, called a *solution of the system*. This solution was discovered through a process called *elimination*.

There are three principal methods of elimination in common use that will now be discussed, and applied in the solution of problems.

126. Elimination by Addition or Subtraction.—The method of elimination shown in § 68 is known as the **method of elimination by addition or subtraction**. The student should now study again the rule on page 110. The following exercises are given for review of that process.

The student should check every answer by seeing if the values of the unknown numbers found satisfy both of the given equations.

EXERCISES

Eliminate by addition or subtraction, solve, and check:

1. $\begin{cases} 3a - b = 21, \\ 2a + b = 4. \end{cases}$

4. $\begin{cases} 7s + t = 42, \\ 3s - t = 8. \end{cases}$

2. $\begin{cases} m + 4n = 4, \\ m - 2n = 16. \end{cases}$

5. $\begin{cases} 4w + v = 25, \\ 5w - 2v = 35. \end{cases}$

3. $\begin{cases} 2x - y = 5, \\ 5x - 2y = 14. \end{cases}$

6. $\begin{cases} 4A - 3B = 1, \\ 3A - 4B = 6. \end{cases}$

- | | |
|--|---|
| 7. $\begin{cases} 7x = 2y - 15, \\ 2x = y - 6. \end{cases}$ | 15. $\begin{cases} m = 4 - 2n, \\ 2n + 12 = m. \end{cases}$ |
| 8. $\begin{cases} 3p + 2q = 12, \\ 4p = 3q - 1. \end{cases}$ | 16. $\begin{cases} 8W + w = 7, \\ 11W + 2w = 28. \end{cases}$ |
| 9. $\begin{cases} 5R - 2r = 1, \\ 8R = 5r - 11. \end{cases}$ | 17. $\begin{cases} 10f + 4t = 22, \\ t + 5f = 11. \end{cases}$ |
| 10. $\begin{cases} 3M - 2N + 4 = 0, \\ 2M - N + 1 = 0. \end{cases}$ | 18. $\begin{cases} 2x_1 - x_2 = 9, \\ 5x_1 - 3x_2 = 14. \end{cases}$ |
| 11. $\begin{cases} 2h + k = 35, \\ 5h - 3k = 27. \end{cases}$ | 19. $\begin{cases} \frac{a}{2} + \frac{b}{7} = 9, \\ \frac{a}{7} - \frac{b}{2} + 5 = 0. \end{cases}$ |
| 12. $\begin{cases} 5y - 5x = 15, \\ 3x + 5y = 71. \end{cases}$ | |
| 13. $\begin{cases} 3r_1 + 2r_2 = 4, \\ 4r_2 - 3r_1 + 1 = 0. \end{cases}$ | 20. $\begin{cases} \frac{w}{6} + \frac{v}{4} = 6, \\ \frac{w}{4} + \frac{v}{6} = 5\frac{2}{3}. \end{cases}$ |
| 14. $\begin{cases} -4B + 3C = 45, \\ 2C + 6B = 4. \end{cases}$ | |

127. Elimination by Comparison. — The method of elimination by comparison is illustrated in the following example.

EXAMPLE. — Solve $\begin{cases} 4x + y = 34, \\ 4y + x = 16. \end{cases}$ (1)

Eliminate x .

Solving (1) for x , $x = \frac{34 - y}{4}$. (3)

Solving (2) for x , $x = 16 - 4y$. (4)

Comparing the two values of x given in (3) and (4),

$$\frac{34 - y}{4} = 16 - 4y. \quad (5)$$

Solving (5) for y , $y = 2$.

Replacing y in (2) by its value 2,

$$8 + x = 16.$$

Solving, $x = 8$.

Hence the solution is $x = 8, y = 2$.

This example illustrates the rule:

Solve each of the equations for the value of either one of the unknown numbers, expressed in terms of the other, and write one of the

values equal to the other. Solve the resulting equation. Substitute the value found for the one unknown number in either of the original equations, and solve for the other unknown number.

EXERCISES

Eliminate by comparison, solve, and check:

1. $\begin{cases} m + 2n = 8, \\ m - n = -1. \end{cases}$
2. $\begin{cases} x - 5a = 7, \\ 2x - 15a = 9. \end{cases}$
3. $\begin{cases} r + 2s = 4, \\ 3r - s = 5. \end{cases}$
4. $\begin{cases} 2r_1 - r_2 = 9, \\ 5r_1 - 3r_2 = 14. \end{cases}$
5. $\begin{cases} 3A + 2B = 2, \\ A - B = 9. \end{cases}$
6. $\begin{cases} 3p - q = 1, \\ 2p + 5q = 41. \end{cases}$
7. $\begin{cases} v + 2t = 4, \\ 2t - v + 12 = 0. \end{cases}$
8. $\begin{cases} 9M - 5N = 13, \\ 5M + N = 11. \end{cases}$
9. $\begin{cases} 3e - 5f = 5, \\ 7e + f = 265. \end{cases}$
10. $\begin{cases} 2R - T = 5, \\ R + 2T = 25. \end{cases}$
11. $\begin{cases} 10V + 3n = 174, \\ 3V + 10n = 125. \end{cases}$
12. $\begin{cases} 5x + 2x' = 1, \\ 13x + 8x' = 11. \end{cases}$
13. $\begin{cases} 2h + k = 9, \\ 5h + 3k = 25. \end{cases}$
14. $\begin{cases} B + 2Q + 2 = 0, \\ 4B - 7Q = 37. \end{cases}$
15. $\begin{cases} 6y - 5z = 1, \\ 9y + 10z = 12. \end{cases}$
16. $\begin{cases} 7L + 3Q + 9 = 0, \\ 6Q - 9L = 28. \end{cases}$
17. $\begin{cases} 3W + 2D = 17, \\ 4W + D = 16. \end{cases}$
18. $\begin{cases} 8a + 5t = 5, \\ 3a - 2t = 29. \end{cases}$
19. $\begin{cases} 5x + 9y = 8, \\ 6y - 9x + 7 = 0. \end{cases}$
20. $\begin{cases} 5V + 4P = 22, \\ 3V + P = 9. \end{cases}$
21. $\begin{cases} \frac{T}{6} + \frac{5}{3} = t, \\ \frac{2T}{5} + 3 = \frac{7t}{5}. \end{cases}$
22. $\begin{cases} \frac{R}{3} - \frac{r}{2} = \frac{19}{15}, \\ \frac{7R}{4} + r = \frac{1}{2}. \end{cases}$
23. $\begin{cases} W - \frac{2}{3} + \frac{7w}{3} = 0, \\ \frac{7W}{2} + 1 + 4w = 0. \end{cases}$

128. Elimination by Substitution. — The method of elimination by substitution is illustrated in the following example.

EXAMPLE. — Solve $\begin{cases} 4w - 5v = 26, \\ 3w - 6v = 15. \end{cases}$ (1)

Eliminate w .

Solving (1) for w , $w = \frac{26 + 5v}{4}$. (3)

Substituting $\frac{26 + 5v}{4}$ in place of w in (2),

$$3\left(\frac{26 + 5v}{4}\right) - 6v = 15. \quad (4)$$

Solving (4), $v = 2$.

Replacing v by its value 2 in (1),
 $4w - 10 = 26$.

Solving, $w = 9$.

Hence, the solution is $w = 9$, $v = 2$.

This example illustrates the rule:

Solve one of the equations for the value of either of the unknown numbers, expressed in terms of the other, and substitute this value in place of that number in the other equation. Solve the resulting equation. Substitute the value found for the one unknown number in either of the original equations, and solve for the other unknown number.

EXERCISES

Eliminate by substitution, solve, and check:

1. $\begin{cases} a + b = 5, \\ a + 3b = 9. \end{cases}$

2. $\begin{cases} x - y = 4, \\ x + 2y = 16. \end{cases}$

3. $\begin{cases} m + 4n = 7, \\ m + 6n = 9. \end{cases}$

4. $\begin{cases} x_1 + x_2 = 30, \\ 3x_1 - 2x_2 = 25. \end{cases}$

5. $\begin{cases} R - S = 4, \\ 2R + S = 14. \end{cases}$

6. $\begin{cases} 2V + t = 7, \\ V + 3t = 16. \end{cases}$

7. $\begin{cases} 3a - 7z = 40, \\ 4a - 3z = 9. \end{cases}$

8. $\begin{cases} 4p - 5q = 26, \\ 3p - 6q = 15. \end{cases}$

9. $\begin{cases} 3D + 7Q = 16, \\ 2D + 5Q = 13. \end{cases}$

10. $\begin{cases} 3g + 2k = 26, \\ 5g - 2k = 38. \end{cases}$

11. $\begin{cases} 7A - 9B = 13, \\ 5A + 2B = 10. \end{cases}$
12. $\begin{cases} 7r_1 + 4r_2 = 1, \\ 9r_1 + 4r_2 = 3. \end{cases}$
13. $\begin{cases} T + t = 10, \\ T - t = 4. \end{cases}$
14. $\begin{cases} 7v + s = 42, \\ 3v - s = 8. \end{cases}$
15. $\begin{cases} 2M - b = 5, \\ 5M - 2b = 14. \end{cases}$
16. $\begin{cases} A - 8p = 45, \\ 3A - p = 20. \end{cases}$
17. $\begin{cases} 3x + 2y = 26, \\ 5x = 38 + 2y. \end{cases}$
18. $\begin{cases} 7m = 26 + 3a, \\ 6a = 6m - 20. \end{cases}$
19. $\begin{cases} 8y + d = 60, \\ 7y = 9 - 20d. \end{cases}$
20. $\begin{cases} 16 - 7t = 3r, \\ 2r - 13 + 5t = 0. \end{cases}$
21. $\begin{cases} a + k = 30, \\ \frac{a}{2} = \frac{25}{3} + \frac{2k}{3}. \end{cases}$
22. $\begin{cases} \frac{2}{a-5} + \frac{5}{b-4} = 0, \\ 3a = 2b + 7. \end{cases}$
23. $\begin{cases} \frac{2w}{3} - v = 5, \\ \frac{4w + v}{6w + v} = \frac{2}{5}. \end{cases}$

MISCELLANEOUS EXERCISES

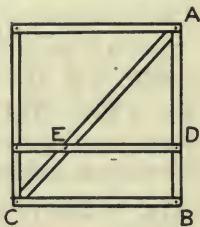
Eliminate by the method that seems best suited, and solve:

1. $\begin{cases} x + y = 10, \\ x - y = 4. \end{cases}$
2. $\begin{cases} 8v + w = 60, \\ 7v - 10w = 9. \end{cases}$
3. $\begin{cases} 21R + 8t + 66 = 0, \\ 49R + 53 = 15t. \end{cases}$
4. $\begin{cases} 5M + 63 = 2b, \\ 2M + b = 18. \end{cases}$
5. $\begin{cases} 2a - 2b = 6\frac{2}{3}, \\ 7a = 26 + 3b. \end{cases}$
6. $\begin{cases} \frac{1}{3}p - \frac{1}{4}q = 2, \\ \frac{1}{4}p - \frac{1}{2}q = 7. \end{cases}$
7. $\begin{cases} 3n + 15z = 7, \\ 12 - n + 5z = 0. \end{cases}$
8. $\begin{cases} \frac{5L}{6} - 7 + \frac{d}{4} = 0, \\ 3 + \frac{d}{8} = \frac{2L}{3}. \end{cases}$
9. $\begin{cases} \frac{k}{3} = 9m - 5\frac{3}{4}, \\ 2\frac{1}{4} - \frac{7m}{3} = \frac{3k}{4}. \end{cases}$
10. $\begin{cases} \frac{1}{x} + \frac{3}{y} = \frac{1}{xy}, \\ 2x - 3y = 1. \end{cases}$
11. $\begin{cases} 2 - 5P = 4V, \\ \frac{3}{P-1} = \frac{4}{V+2}. \end{cases}$

129. Problems solved by Systems of Equations. — For the steps in the process of expressing a problem by means of a system of equations with two unknown numbers see § 69. In the following problems use that method of elimination which seems best suited to each individual problem.

EXERCISES

- Find two numbers whose sum is 29, and difference 13.
- Find two numbers whose sum is 9, and such that one of them exceeds twice the other by 27.
- The difference between two numbers is 72, and one of them is 4 times the other. What are the numbers?
- The sum of two numbers is 64, and their quotient 7. What are the numbers?
- The difference between two numbers is 21, and if the larger is divided by the smaller, the quotient is 4 and the remainder 3. Find the numbers.
- The value of a fraction is $\frac{2}{3}$, and if 7 be added to each term, the value of the resulting fraction is $\frac{3}{4}$. Find the fraction.
- A board 10 ft. long is to be cut into two pieces whose lengths are as 3 and 4. How long are the pieces?
- A surveyor wishes to set a stake in a line 320 ft. long, so that its distances from the ends of the line are as 7 is to 19. Where must he place it?
- In building a bridge a steel bar 12 ft. long is to be bolted to another bar at a point which divides the 12 ft. bar into parts whose lengths are as 7 and 17. Where must the hole be bored for the bolt?
- In the steel frame in the figure the brace AC is 15 ft., AD is 7 ft., and DB is 3 ft. It is desired to find where to bore the hole for the bolt E in AC . By proportion, $\frac{AE}{EC} = \frac{AD}{DB}$. Find where to locate E .



X 11. Two boys desired to weigh themselves. They had no scales, but had a 10 lb. weight. They placed a board across a support, as in playing teeter. When they sat, one 5 ft. from the fulcrum and the other 4 ft. from it, they balanced. When the smaller boy took the 10 lb. weight in his hands, he had to sit only 4 ft. 4 in. from the fulcrum to balance. What were their weights?

12. Two weights balance when the larger is 3 ft. and the smaller 4 ft. from the fulcrum. If the smaller weight, with 3 lb. added to it, is moved to a position 3 ft. $5\frac{1}{4}$ in. from the fulcrum, they still balance. Find the weights.

13. Two weights balance when one is 4 in. and the other 6 in. from the fulcrum. If the first weight is decreased 12 lb., the other must be moved 1 in. nearer the fulcrum to balance. Find the weights.

14. Two weights balance when one is 3 ft. and the other 2 ft. from the fulcrum. The larger weight is decreased 12 oz., and the smaller weight is decreased 6 oz. and moved $2\frac{2}{3}$ in. nearer the fulcrum. Again they balance. Find the weights.

15. Two partners in business, A and B, are to divide profits in the ratio of 2 to 5. In dividing a profit of \$6300, how much should each receive?

16. A man invests part of \$12,000 at 5% and the balance at 6%. His annual income from both investments is \$645. Find the number of dollars in each investment.

17. A man invests \$2000, part at 5% and the rest at 4%. The annual income from the 5% investment exceeds that of the 4% investment by \$2.80. Find the amount of each investment.

18. An investor purchased two kinds of bonds. One kind yielded 3% and the other 4%. His yearly income from both was \$558. Had he invested as much in 4% bonds as in 3% bonds, and *vice versa*, his yearly income would have been \$576. How much did he invest in each kind of bonds?

19. A man invested in \$22,500 worth of bonds, part of them New York City 4s and part U. S. Steel 5s. His annual income

was \$1045. Had he purchased as many New York City 4s as he did U. S. Steel 5s, and *vice versa*, his annual income would have been only \$980. What was the value of each kind of bonds bought?

20. I change \$3 into dimes and nickels. There are 50 coins in all. How many dimes and how many nickels are there?

21. I got \$100 changed into \$5 bills and \$10 bills. I received 16 bills in all. How many bills of each kind did I get?

22. If a merchant blends 25-cent coffee and 32-cent coffee to sell at 30 cents a pound, what quantities of each grade of coffee must he take to make 50 lb. of the blend?

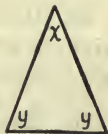
23. How much each of copper, specific gravity 8.9, and zinc, specific gravity 6.9, must be combined to produce 1000 cu. cm. of brass, specific gravity 8.4? (See Problem 7, page 112.)

24. In a number of two digits the sum of the digits is 11, and when the digits are interchanged, the number is diminished by 45. Find the number.

25. Two angles are called supplementary when their sum is 180° . If one of two supplementary angles is 20° more than twice the other, how many degrees in each?

26. The sum of the angles of a triangle is always 180° . If each of the angles at the base of the triangle is twice the angle at the vertex, how many degrees in each angle of the triangle?

27. The altitude of a trapezoid is 6 ft. and its area 84 sq. ft. One base is 8 ft. longer than the other. Find the lengths of the bases.



28. If the length of a rectangle is diminished 3 in. and its width increased 2 in., its area is unchanged; and if its length is increased 5 in. and its width diminished 2 in., its area is unchanged. Find the length and width.

29. Two groups of students of surveying were sent to find the area of a rectangular field. One group got the length 4 ft. too small and the width 6 ft. too large, which gave the area 2960 sq.

ft. too large. The other group got the length 3 ft. too large and the width 3 ft. too small, which gave the area 741 sq. ft. too small. Find the dimensions of the field.

30. The circumference of the fore wheel of a carriage is 3 ft. less than that of the rear wheel. One goes the same distance in 40 revolutions that the other does in 30. Find the circumference of each.

31. A belt runs over two pulleys. One makes 10 revolutions while the other makes 3. If the larger pulley were replaced by one whose circumference was 20 in. less, it would only make 10 revolutions while the other made 4. Find the circumference of each pulley.

32. A man who can row 6 mi. an hour downstream can row 2 mi. an hour upstream. What is the speed of the current?

33. A crew rows downstream $6\frac{1}{2}$ mi. in an hour, and returns in 4 hr. 20 min. What is the speed of the current, and at what rate could the crew row in still water?

34. An aviator flying against the wind makes 20 mi. an hour. On returning with the wind he makes 50 mi. an hour. What is the speed of the wind, and what would be his speed if there were no wind?

35. If an aviator flies to a point 24 mi. distant, against the wind, in 40 min., and returns in 30 min., what is the speed of the wind, and what would be his speed if there were no wind?

36. Two trains, each 300 ft. long, run on parallel tracks. If running in the same direction, it requires 20 sec. for one to pass the other. If running in opposite directions, it requires only 4 sec. for them to pass. What are the speeds of the trains?

37. A passenger train and a freight train run in opposite directions on parallel tracks. The speed of the passenger train is 60 ft. per second, and the speed of the freight train is 40 ft. per second. The freight train is 180 ft. longer than the passenger train. It takes the trains 12.6 sec. to pass. Find the length of each train.

38. A passenger and a freight train run on parallel tracks and in the same direction. The speed of the passenger train is 45 mi. an hour, and the speed of the freight train is 25 mi. an hour. The freight train is 96 ft. longer than the passenger train. It takes the passenger train 36 sec. to pass the freight. Find the length of each train.

39. How much milk testing 4% butter fat and cream testing 24% butter fat must be mixed to make 20 gal. that test 20% butter fat?

40. How much milk testing 3.8% butter fat and cream testing 26.6% butter fat must be mixed to make 16 gal. that test 18% butter fat?

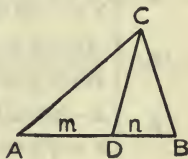
41. Fifteen pounds of tin weigh 13 lb. in water and 15 lb. of zinc weigh 13.5 lb. in water. How much tin and how much zinc in an alloy which weighs 56 lb. in air and 49 lb. in water?

42. Some authorities claim that the daily ration for the average adult workingman should contain 4 oz. of protein and an equal amount of fat. White bread contains 9% protein and 1% fat. Mutton contains 14% protein and 37% fat. Find how many ounces each of bread and mutton would be required to make a daily ration.

43. Eggs contain 13% protein and 9% fat. From the facts given in Problem 42, can bread and eggs be used to make a standard ration?

44. A society that wished to raise \$100 gave an entertainment. They estimated that enough children and adults would attend that they could raise this amount by charging children 10 cents and adults 25 cents admission, and that if they charged children 15 cents and adults 30 cents admission, they would raise \$26 more than the amount desired. How many children and how many adults did they expect would attend?

45. In the triangle ABC , the line CD bisects (divides into two equal parts) the angle at C . It is known from geometry that CD divides AB



into two parts, m and n , which are proportional to the adjacent sides. If $AC = 15$ in., $BC = 12$ in., and $AB = 10$ in., find the lengths of m and n .

130. Systems of Literal Equations. — Systems of linear equations in which one or more of the known numbers are literal are solved by the methods of §§ 126, 127, 128, the solutions being expressed in terms of these literal quantities.

$$\text{EXAMPLE. — Solve for } x \text{ and } y : \begin{cases} ax - by = a^3, \\ bx - ay = b^3. \end{cases} \quad (1)$$

$$\text{Multiplying (1) by } b, \quad abx - b^2y = a^3b. \quad (2)$$

$$\text{Multiplying (2) by } a, \quad abx - a^2y = ab^3. \quad (3)$$

$$\text{Subtracting (4) from (3),} \quad a^2y - b^2y = a^3b - ab^3. \quad (4)$$

$$\text{Factoring (combining terms),} \quad (a^2 - b^2)y = ab(a^2 - b^2).$$

$$\text{Dividing by } a^2 - b^2, \quad y = ab.$$

$$\text{Substituting } ab \text{ for } y \text{ in (1),} \quad ax - ab^2 = a^3.$$

$$ax = a^3 + ab^2.$$

$$x = a^2 + b^2.$$

$$\text{Hence, the solution is} \quad x = a^2 + b^2, \quad y = ab.$$

EXERCISES

Solve for x and y :

$$1. \begin{cases} ax + by = a^2 + b^2, \\ bx + ay = 2ab. \end{cases}$$

$$2. \begin{cases} ax + by = a^2 - b^2, \\ ay + bx = a^2 - b^2. \end{cases}$$

$$3. \begin{cases} my + x = m + n, \\ mx + my = m^2 + n. \end{cases}$$

$$4. \begin{cases} x + Ry = S, \\ 2x - Sy = R. \end{cases}$$

$$5. \begin{cases} Ax + y = B, \\ Bx + y = A. \end{cases}$$

$$6. \begin{cases} x - p^2y = 0, \\ x + q^2y = 1. \end{cases}$$

$$7. \begin{cases} r_1x + r_2y = 1, \\ r_2x + r_1y = 1. \end{cases}$$

$$8. \begin{cases} ax - b^2y = a^3 - b^3, \\ a^2y - bx = 0. \end{cases}$$

$$9. \begin{cases} 2px + 2qy = 4p^2 + q^2, \\ x - 2y = 2p - q. \end{cases}$$

$$10. \begin{cases} 12x - 11y = R_1 + 12R_2, \\ x + y = 2R_1 + R_2. \end{cases}$$

$$11. \begin{cases} (m + n)x = 1 - py, \\ (m + n)y + px = 1. \end{cases}$$

$$12. \begin{cases} vx - ty = 0, \\ x + y = w. \end{cases}$$

$$13. \begin{cases} Px - Qy = R, \\ x + y = Q. \end{cases}$$

$$14. \begin{cases} \frac{x}{2} + \frac{y}{3} = a, \\ \frac{x}{3} + \frac{y}{4} = b. \end{cases}$$

$$15. \begin{cases} \frac{x}{a-b} + \frac{y}{a+b} = 2, \\ x + y = 2a. \end{cases}$$

$$16. \begin{cases} \frac{x}{2} - 3y = \frac{m-12n}{4}, \\ 8x + 9y = 4m + 9n. \end{cases}$$

$$17. \begin{cases} \frac{x}{v} + 5 = \frac{2y}{v}, \\ \frac{3x}{v} - \frac{7y}{4v} = \frac{3}{4}. \end{cases}$$

18. If m be added to the numerator of a certain fraction, the value of the resulting fraction is 4. But if n be added to the denominator, the value of the resulting fraction is 3. What is the fraction?

19. The sum of two numbers is s , and the quotient of the first divided by the second is q . Find the numbers.

20. If a boy who weighs W pounds and one who weighs P pounds balance at the ends of a teeter board l feet long, find the lengths of the two parts into which the board is divided at the fulcrum.

21. Part of \$12,000 is invested at $x\%$ and the rest at $y\%$. The annual income from both investments is d dollars. Find the number of dollars in each investment.

22. If milk tests $m\%$ butter fat and cream $n\%$ butter fat, how much of each must be taken to make w pounds of cream testing $a\%$ butter fat?

131. Number of Solutions of a System.—It has been seen that each of the problems in the preceding sections has one and only one solution. That this is true in general of a system of two linear equations with two unknown numbers may be shown as follows:

Any two linear equations whose unknown numbers are x and y may be written in the forms

$$\begin{cases} ax + by = c, \\ dx + ey = f, \end{cases}$$

where a, b, c, d, e , and f represent any numbers whatever. If this general system be solved,

$$x = \frac{ce - bf}{ae - bd}, \quad y = \frac{af - cd}{ae - bd}.$$

Hence, in general, the system has *one and only one solution*, depending upon the values of a, b, c, d, e , and f . But these numbers may be so related that there is *no solution*, or that there is an *infinitely great number of solutions*.

$$\text{EXAMPLE 1. — Solve } \begin{cases} 2x - y = 12, & (1) \\ 6x - 3y = 4. & (2) \end{cases}$$

$$\text{Multiplying (1) by 3, } 6x - 3y = 36. \quad (3)$$

$$\text{Subtracting (2) from (3), } 0 = 32.$$

This is absurd. It is impossible to eliminate one of the unknown numbers without at the same time eliminating the other, and hence no solution can be found. Or, looking at the matter in a different way, the supposition that both equations can be true simultaneously, that is, for the same values of x and y , leads to the conclusion that $0 = 32$, which is false. Hence the supposition is false, and the equations have no common solution.

Two linear equations having no common solution are called **inconsistent equations**.

$$\text{EXAMPLE 2. — Solve } \begin{cases} 2x - y = 3, & (1) \\ 4x = 6 + 2y. & (2) \end{cases}$$

$$\text{Transposing (2) gives } 4x - 2y = 6. \quad (3)$$

$$\text{Dividing (3) by 2, } 2x - y = 3. \quad (4)$$

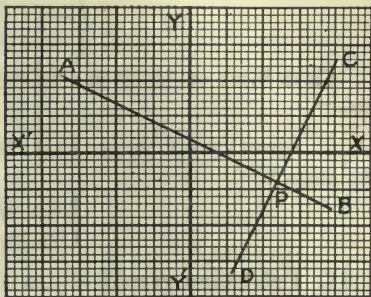
Equation (1) is the same as equation (4). Hence all of their solutions are common.

Two equations, such as (1) and (2) above, that have all of their solutions common are called **equivalent equations**. It is evident that one of the equations may always be derived from the other by transposition, etc.

132. Systems solved Graphically. — The solution of a system of two equations in two unknown numbers may be found graphically, without performing the process of elimination.

$$\text{EXAMPLE. — Solve graphically } \begin{cases} x + 2y = 4, \\ 2x - y = 28. \end{cases}$$

Upon the same pair of axes draw the graphs of both equations, as in § 124. Only two points of each graph need be located in order to draw it. The graph of $x + 2y = 4$ is AB , and the graph of $2x - y = 28$ is CD . The graphs AB and CD meet at a point P . Since P is on AB , the values of x and y corresponding to the point P must satisfy $x + 2y = 4$. And since P is also on CD , the values of x and y corresponding to the point P must also satisfy $2x - y = 28$. Hence the values of x and y corresponding to the point P , which are seen to be $x = 12$, $y = -4$, must be the solution of the system. See if $x = 12$, $y = -4$ satisfies both equations.



To solve a system graphically, draw the graphs of both equations upon the same axes. The set of values of the unknown numbers corresponding to the point where the graphs meet is the solution of the system.

EXERCISES

Solve graphically:

1. $\begin{cases} x + y = 10, \\ x - y = 4. \end{cases}$

2. $\begin{cases} 3x - y = 2, \\ 7x - 2y = 8. \end{cases}$

3. $\begin{cases} 5x + 4y = 42, \\ 6x - y = 4. \end{cases}$

4. $\begin{cases} x + 4y = 10, \\ 3x - y = 12. \end{cases}$

5. $\begin{cases} 4x + y = 3, \\ 2x + 3y = 4. \end{cases}$

6. $\begin{cases} x + y = 5, \\ 2x + 3y = 12. \end{cases}$

7. $\begin{cases} 3x + 2y = 12, \\ 4x - 3y + 1 = 0. \end{cases}$

8. $\begin{cases} 7x - 2y + 15 = 0, \\ y - 2x = 6. \end{cases}$

9. $\begin{cases} 2x + 5y = 15, \\ 3x - 4y = 11. \end{cases}$

10. $\begin{cases} 4x - 3y = 1, \\ 3x - 4y = 6. \end{cases}$

11. $\begin{cases} 2x - y = 5, \\ x + 2y = 25. \end{cases}$

12. $\begin{cases} 2x - y = 5, \\ 5x - 2y = 14. \end{cases}$

13. Show by graphs that in general a system of two equations in two unknown numbers has one and only one solution.

14. Draw the graphs of the inconsistent equations of the system

$$\begin{cases} 2x + y = 11, \\ 4x + 2y = 43. \end{cases}$$

What kind of lines are they? Would you naturally expect this?

15. Show graphically that the following equations are inconsistent:

$$\begin{cases} x - 3y = 6, \\ 3x = 4 + 9y. \end{cases}$$

16. Determine graphically whether or not the following equations are consistent:

$$\begin{cases} 5x - y = 0, \\ 3y = 40 + 15x. \end{cases}$$

17. By drawing their graphs, show that $3x - 2y = 6$, $x + y = 7$, and $4x - y = 13$ have a common solution. What is it?

18. Show by their graphs that $3x - 2y = 12$, $4x - 3y + 1 = 0$, and $2x - y = 11$ have no common solution.

19. Determine graphically whether or not $3x + 2y = 18$, $x - y = 8$, and $4x + y = 4$ have a common solution.

20. Show graphically that in general three or more linear equations containing the same two unknown numbers cannot have a common solution. When would they have a common solution?

133. Systems of Fractional Equations.—As a general rule, in solving a system of fractional equations, it is best first to clear the equations of fractions. *But in certain cases, as when the unknown numbers occur only in monomial denominators, it is best not to clear of fractions before eliminating.* If such equations were cleared of fractions, they usually would not be linear, and also would give solutions which would not satisfy the original system.

EXAMPLE 1.—Solve
$$\begin{cases} \frac{9}{a} - \frac{4}{b} = 2, \\ \frac{9}{a} + \frac{9}{b} = 5. \end{cases} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

Subtracting (1) from (2), $\frac{13}{b} = 3.$

Solving, $b = 4\frac{1}{3}.$

Substituting $4\frac{1}{3}$ for b in (1), $\frac{9}{a} - \frac{4}{4\frac{1}{3}} = 2.$

Solving, $a = 3\frac{3}{4}.$

EXAMPLE 2.—Solve $\begin{cases} \frac{2}{3x} + \frac{3}{2y} = 5, \\ \frac{3}{2x} - \frac{2}{3y} = 3. \end{cases}$ (1)

(2)

Multiplying (1) by $\frac{2}{3}$, to make the fraction in x the same as in (2),

$$\frac{3}{2x} + \frac{27}{8y} = \frac{45}{4}. \quad (3)$$

Subtracting (2) from (3), $\frac{27}{8y} + \frac{2}{3y} = \frac{45}{4} - 3.$

Solving, $y = \frac{97}{198}.$

Substituting $\frac{97}{198}$ for y in (1), $\frac{2}{3x} + \frac{594}{194} = 5.$

Solving, $x = \frac{97}{113}.$

EXERCISES

Solve:

1. $\begin{cases} \frac{1}{m} + \frac{2}{n} = 2, \\ \frac{2}{m} - \frac{2}{n} = 1. \end{cases}$

2. $\begin{cases} \frac{4}{p} + \frac{9}{q} = 1, \\ \frac{3}{p} + \frac{6}{q} = \frac{1}{2}. \end{cases}$

3. $\begin{cases} \frac{1}{A} + \frac{1}{B} = 7, \\ \frac{1}{A} - \frac{1}{B} = 3. \end{cases}$

4. $\begin{cases} \frac{3}{w} + \frac{8}{v} = 3, \\ \frac{15}{w} - \frac{4}{v} = 4. \end{cases}$

5. $\begin{cases} \frac{3}{r_1} + \frac{4}{r_2} = 3, \\ \frac{6}{r_1} - \frac{2}{r_2} = 1. \end{cases}$

6. $\begin{cases} \frac{10}{P} - \frac{6}{Q} + 1 = 0, \\ \frac{15}{P} + \frac{2}{Q} = 4. \end{cases}$

7. $\begin{cases} \frac{6}{R} + \frac{8}{r} = 3, \\ \frac{12}{R} - \frac{20}{r} + 3 = 0. \end{cases}$

8. $\begin{cases} \frac{3}{x_1} + \frac{2}{x_2} + 12 = 0, \\ \frac{4}{x_1} - \frac{3}{x_2} = 1. \end{cases}$

$$9. \begin{cases} \frac{1}{2a} - \frac{2}{5b} = 4, \\ \frac{5}{a} + \frac{6}{b} = 7. \end{cases}$$

$$11. \begin{cases} \frac{1}{4A} + \frac{7}{2B} = \frac{5}{8}, \\ \frac{1}{2A} - \frac{3}{B} + \frac{5}{28} = 0. \end{cases}$$

$$10. \begin{cases} \frac{2}{5t} + \frac{5}{2T} = 7, \\ \frac{5}{t} + \frac{2}{T} = 29. \end{cases}$$

$$12. \begin{cases} \frac{2}{9x} - \frac{1}{y} = 4, \\ \frac{3}{5x} + \frac{1}{2y} = 7. \end{cases}$$

Solve for x and y :

$$13. \begin{cases} \frac{a}{x} + \frac{b}{y} = c, \\ \frac{a'}{x} + \frac{b'}{y} = c'. \end{cases}$$

$$14. \begin{cases} \frac{1}{mx} + \frac{1}{ny} = 1, \\ \frac{2}{mx} - \frac{3}{ny} = 1. \end{cases}$$

$$15. \begin{cases} \frac{m}{nx} + \frac{n}{my} = \frac{m+n}{mn}, \\ \frac{n}{mx} - \frac{m}{ny} = \frac{n^3 - m^3}{m^2n^2}. \end{cases}$$

16. A water tank can be filled by two pipes in $8\frac{8}{9}$ minutes. If the first is left open 10 minutes and the second 8 minutes, the tank will be filled. In what time can each pipe alone fill the tank?

17. Two steam pumps together can fill a reservoir with water in $6\frac{3}{4}$ hr. If one pump works 8 hr. and the other 3 hr. 48 min., the reservoir will be filled. How long would it take each pump alone to fill it?

18. In a factory which operates two sizes of machines it is found that 2 large machines and 5 small ones can turn out a certain quantity of goods in 12 hr., while 4 large ones and 3 small ones can do it in 10 hr. Find how long it would require one machine of each size alone to turn out the goods.

19. A and B together can do a piece of work in $13\frac{1}{3}$ days. After they have worked 6 days B leaves, and A finishes the work in $16\frac{1}{2}$ days more. In how many days could each of them alone do the work?

20. A current of electricity flows through two branches of a circuit. The total resistance to the current is found to be 40 ohms. When the resistance of one branch is made three times as great, the resistance of the whole circuit is doubled. If r_1 and r_2 are the resistances of the branches, it is known that $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{40}$ and $\frac{1}{3r_1} + \frac{1}{r_2} = \frac{1}{80}$. Find the resistance of each branch.

134. **Systems involving Three Unknown Numbers.**—Some problems involving three unknown quantities may be solved by first expressing them by means of a system of three linear equations containing three unknown numbers. To solve such a system we first eliminate one of the unknown numbers from any two of the equations, then eliminate the same number from one of these two equations and the third equation. This will give rise to two new equations which contain only two unknown numbers. These two equations may then be solved as a new system by the methods of the preceding sections.

$$\begin{array}{ll} \text{EXAMPLE.—Solve} & \begin{cases} x + 2y + 2z = 11, & (1) \\ 2x + y + z = 7, & (2) \\ 3x + 4y + z = 14. & (3) \end{cases} \end{array}$$

By subtraction, eliminate x between (1) and (2).

$$\text{This gives} \quad 3y + 3z = 15. \quad (4)$$

By subtraction, eliminate x between (1) and (3).

$$\text{This gives} \quad 2y + 5z = 19. \quad (5)$$

Eliminating y between (4) and (5),

$$z = 3.$$

By substituting 3 for z in (5) we get

$$y = 2.$$

Now, substituting the values of both y and z in (1), we get

$$x = 1.$$

From a system of four linear equations with four unknown numbers we can, in like manner, derive a new system of three equations with three unknown numbers; and so on.

EXERCISES

Solve:

$$1. \begin{cases} a + b + c = 9, \\ 2a + b - c = 0, \\ 3a - b + c = 5. \end{cases}$$

$$2. \begin{cases} P - 2Q + R = 6, \\ P + 3Q + 2R = 13, \\ 2P - Q + R = 13. \end{cases}$$

$$3. \begin{cases} 2u + 2v + 3w = 4, \\ 3u + 4v + 6w = 7, \\ u + 2v + 6w = 4. \end{cases}$$

$$4. \begin{cases} 5r_1 + 6r_2 + 7r_3 = 3, \\ 10r_1 - 12r_2 + 21r_3 = 3, \\ 15r_1 - 6r_2 + 14r_3 = 4. \end{cases}$$

$$5. \begin{cases} A + B = 1, \\ B + C = 9, \\ A + C + 6 = 0. \end{cases}$$

$$6. \begin{cases} 2x + y - z = 2, \\ x - 2y - 3z = 6, \\ y - z + 1 = 0. \end{cases}$$

$$7. \begin{cases} 2h + k = 7 + l, \\ k = 1 + h, \\ l - k = 1. \end{cases}$$

$$8. \begin{cases} 2p - 2q + 3r = 10, \\ 3p + q - r = 5, \\ p - q + 2r = 7. \end{cases}$$

$$9. \begin{cases} \frac{a}{5} + \frac{b}{2} + \frac{c}{3} = 19, \\ \frac{a}{10} - \frac{b}{10} + \frac{c}{6} = 6, \\ \frac{a}{4} + \frac{b}{5} - \frac{c}{15} = 5. \end{cases}$$

$$10. \begin{cases} \frac{1}{x} + \frac{1}{y} = 2, \\ \frac{1}{x} + \frac{1}{z} = 3, \\ \frac{1}{y} + \frac{1}{z} = 4. \end{cases}$$

$$11. \begin{cases} \frac{2}{3m} + \frac{1}{n} + \frac{3}{10} = 0, \\ \frac{1}{m} + \frac{4}{5p} = \frac{1}{12}, \\ \frac{1}{p} + \frac{7}{30} = \frac{3}{4n}. \end{cases}$$

12. The perimeter of a triangle is 28 in. Two of the sides are equal, and their sum exceeds the third side by 4 in. Find the lengths of the sides.

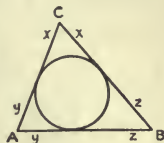
13. The sum of the two sides of a triangle which meet at one vertex is 44 in., at another vertex 40 in., and at the third vertex 36 in. Find the lengths of the three sides.

14. The sum of the three angles of any triangle is 180° . If two angles of a triangle are equal, and each is 5° more than the third, find the size of each.

15. In triangle PQR , angle P is 16° less than angle Q , and angle Q is 4° less than angle R . How many degrees in each?

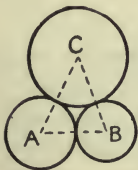
16. It is known in geometry that two tangents drawn from the same point to a circle are equal. Hence, if a circle is inscribed in a triangle, as in the figure, the points of contact divide the sides of the triangle into three sets of equal tangents.

If $AB = 12$ in., $BC = 16$ in., and $AC = 14$ in., show that $x + y = 14$, $x + z = 16$, and $y + z = 12$. Find the distances of the points of contact of the circle with the three sides from the vertices.



17. The sides of a triangle are 24 in., 32 in., and 38 in. Find the points where the inscribed circle touches the sides.

18. Three circles are to be drawn tangent to each other, and with their centers at three given points A , B , and C , respectively. If the distance between A and B is 10 in., between B and C 13 in., and between A and C 15 in., find the radii of the circles.



19. In accurate tool work where holes are to be bored close together in a metal plate, the centers of the holes are first marked carefully to thousandths of an inch. This may be done by first turning out disks on a lathe of such sizes that when placed tangent to each other their centers mark the positions of the centers of the required holes. These disks are then fastened on the metal plate in tangent positions, and the holes bored at their centers.

Three holes are to be bored, the distances between whose centers shall be 0.560 in., 0.620 in., and 0.844 in., respectively. Find the radii of the required disks. See Problem 18.

20. A certain number consists of three digits whose sum is 9. If 198 be subtracted from the number, the remainder will consist of the same digits in a reverse order; and if the number be divided by the hundreds' digit, the quotient will be 108. What is the number?

21. It is given that the standard daily ration for an adult laboring man should include 16 oz. starch, 4 oz. fat, and 4 oz. albumen. The amount of these materials in bread, butter, and beef are as follows:

FOOD	STARCH	FAT	ALBUMEN
Bread	54 %	1 %	9 %
Butter	0 %	83 %	1 %
Beef	0 %	15 %	15 %

Find the quantities of these foods required to make a daily ration.

22. Potatoes contain 20 % starch, 1 % fat, 2 % albumen ; pork contains no starch, 26 % fat, 13 % albumen ; and cream contains 4 % starch, 18 % fat, 2 % albumen. Find the quantities of these foods required to make a standard daily ration.

23. There are three compounds composed of different metals. The first contains 7 parts (in weight) silver, 3 parts copper, and 6 parts tin ; the second contains 12 parts silver, 3 parts copper, and 1 part tin ; the third contains 4 parts silver, 7 parts copper, and 5 parts tin. How much of each of these three compounds must be taken in order to form a fourth which shall contain 8 oz. of silver, $3\frac{3}{4}$ oz. of copper, and $4\frac{1}{2}$ oz. of tin ?

24. In an athletic contest team A won with a total of 35 points, team B got second place with a total of 32 points, and team C got third place with a total of 31 points, as shown in the table:

TEAM	1sts	2ds	3ds	TOTAL
A	7	3	1	35
B	5	4	4	32
C	2	7	9	31

How many points does each place count ?

SUPPLEMENTARY EXERCISES

Solve:

$$1. \begin{cases} \frac{5M}{6} + \frac{N}{4} = 7, \\ \frac{2M}{3} - \frac{N}{8} = 3. \end{cases}$$

$$2. \begin{cases} \frac{4W+w}{6W+w} = \frac{2}{5}, \\ 5+w = \frac{2W}{3}. \end{cases}$$

$$3. \begin{cases} \frac{9r_1+4}{r_1+r_2+1} = 4, \\ \frac{r_1+3}{r_1+1} = \frac{r_2+1}{r_2-1}. \end{cases}$$

$$4. \begin{cases} \frac{1}{a+b} + \frac{1}{a-b} = 0, \\ \frac{a-3b}{b-2a} = 2. \end{cases}$$

$$5. \begin{cases} \frac{1}{1+x} + \frac{1}{1+y} = 1, \\ \frac{2}{1+x} - \frac{1}{1+y} = \frac{1}{2}. \end{cases}$$

$$6. \begin{cases} \frac{4}{x-2} + \frac{3}{y+1} = 1, \\ \frac{1}{x-2} + \frac{6}{y+1} = \frac{9}{8}. \end{cases}$$

Solve for x and y :

$$7. \begin{cases} \frac{a}{x} - \frac{b}{y} = c, \\ \frac{2a}{3x} + \frac{3b}{4y} = 2c. \end{cases}$$

$$8. \begin{cases} \frac{3n}{x} + \frac{n-1}{y} = m, \\ \frac{m}{2x} - \frac{m+1}{y} = n. \end{cases}$$

Solve:

$$9. \begin{cases} a+b+c=2, \\ b+c+d=4, \\ c+d+a=6, \\ a+b+d=3. \end{cases}$$

$$10. \begin{cases} r+s+t=18, \\ r+s+w=18, \\ s+w+t=18, \\ r+w+t=18. \end{cases}$$

$$11. \begin{cases} x+y+z+w=10, \\ x-y-z+w=0, \\ 2x+y+3z-w=9, \\ 3x-y+z-2w+4=0. \end{cases}$$

$$12. \begin{cases} 2p-q-r+s=13, \\ p+q-r-s+1=0, \\ p-q+r-s+5=0, \\ 3p+2q-r+2s=17. \end{cases}$$

13. Find a number of three digits, such that the sum of the digits shall be 15; the sum of the hundreds' digit and the ones' digit shall be 1 less than the tens' digit; and the ones' digit subtracted from 4 times the hundreds' digit shall equal the tens' digit.

14. A merchant has two kinds of tea. If he mix 3 pounds of the poorer with 7 pounds of the better, the mixture will be worth $76\frac{1}{2}\text{¢}$ a pound; but if he mix 7 pounds of the poorer with 3 pounds of the better, the mixture will be worth $68\frac{1}{2}\text{¢}$ a pound. What is the price of each kind of tea?

15. There are two alloys of copper and silver, of which one contains 3 times as much copper as silver, and the other contains 5 times as much silver as copper. How much must be taken of each alloy to make 7 pounds, of which half shall be silver and the other half copper?

16. If the altitude of a rectangle be increased 4 inches, and its base diminished 2 inches, the area will be increased 22 square inches; and if the altitude be increased 1 inch, and the base diminished 1 inch, the area will be increased 2 square inches. Find the base and the altitude of the rectangle.

17. A company of men rented a yacht. When they paid their rental they found that if there had been 2 more persons to pay the same bill, each would have paid 50 cents less than he did; and if there had been 2 fewer persons, each would have paid \$1 more than he did. Find the number of persons and the amount that each paid.

CHAPTER XIII

SQUARE ROOT. QUADRATIC SURDS

135. Square Roots of Polynomials.—Since $(a+b)^2 = a^2 + 2ab + b^2$,

$$\sqrt{a^2 + 2ab + b^2} = a + b.$$

NOTE.—As shown in § 72, every quantity has two square roots, differing only in sign. Thus, the other square root of $a^2 + 2ab + b^2$ is $-(a + b)$, or $-a - b$. In this chapter we shall consider only one of the square roots of any expression, viz. the one of which the first term is positive.

A study of the above identity will reveal the process of finding the square root of any polynomial which is a perfect square.

If the polynomial has three terms arranged according to the powers of one letter, these correspond to the terms of $a^2 + 2ab + b^2$. Evidently, the first term a of the root is a square root of a^2 . If a^2 be subtracted from the trinomial, the remainder is $2ab + b^2$. The second term of the root, b , may be found by dividing $2ab$, the first term of the remainder, by $2a$, or twice the term of the root already found.

The work is usually arranged as follows :

$$\begin{array}{r} a^2 + 2ab + b^2 \overline{)a + b} \\ \underline{a^2} \\ 2a \overline{)2ab + b^2} \\ \underline{2a + b} \\ 2ab + b^2 \end{array}$$

The divisor $2a$, used in finding the second term, is called the **trial divisor**. When the second term b is added to the trial divisor, it gives the **true divisor**, $2a + b$, because when the latter is multiplied by b it gives the entire remainder $2ab + b^2$.

NOTE. — Care must be taken to first arrange the terms in any problem in descending or ascending powers of some letter. Each remainder and each divisor must also be arranged like the original expression.

The above examples illustrate the general rule:

(1) Write the given polynomial in descending or ascending powers of some letter.

(2) Take the square root of the first term for the first term of the root, and subtract the first term from the polynomial.

(3) Double the root found for the first trial divisor, divide the first term of the remainder by this, and write the quotient as the second term of the root.

(4) Add this quotient to the trial divisor to obtain the true divisor, multiply the true divisor by the second term of the root, and subtract the product from the preceding remainder.

(5) If there is still a remainder, double all of the root already found, for a new trial divisor, and proceed as before. Continue this process until all terms of the root are found.

EXERCISES

Find the square roots of the following:

1. $4a^2 + 20a + 25$.
2. $1 - 16y + 64y^2$.
3. $m^4 + 25n^2 - 10m^2n$.
4. $5x^2 + x^4 - 2x^3 + 4 - 4x$.
5. $4t^4 + 49 - 3t^2 - 70t + 20t^3$.
6. $1 + 2b - b^2 + 3b^4 - 2b^5 + b^6$.
7. $49A^8 + 42A^6 - 19A^4 - 12A^2 + 4$.
8. $r^4 + 21r^2 + 4 - 10r^3 + 20r$.
9. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.
10. $4w^4 - 4w^3 + 17w^2 - 8w + 16$.
11. $13M^2N^2 + M^4 - 6M^3N - 12MN^3 + 4N^4$.
12. $x^4 - 2x^3y - 2xy^3 + 3x^2y^2 + y^4$.
13. $9p^4 - 12p^3q - 26p^2q^2 + 20pq^3 + 25q^4$.

$$14. 49 - 42 V + 37 V^2 + 4 V^4 - 12 V^3.$$

$$15. x^6 - 2 x^5 + 3 x^4 - 4 x^3 + 3 x^2 - 2 x + 1.$$

$$16. m^6 - 6 m^5 n + 15 m^4 n^2 - 20 m^3 n^3 + 15 m^2 n^4 - 6 m n^5 + n^6.$$

$$17. v^6 + 16 v^6 - 4 v^5 w + 10 v^4 w^2 - 24 v w^5 - 20 v^3 w^3 + 25 v^2 w^4.$$

$$18. 4 P^6 - 20 P^5 + 46 P^2 + 41 P^4 + 9 - 52 P^3 - 24 P.$$

136. Square Roots of Arithmetical Numbers.—Any arithmetical number of two or more figures is in nature a polynomial.

$$\begin{aligned}\text{Thus, } 5263 &= 5000 + 200 + 60 + 3 \\ &= 5 \times 10^3 + 2 \times 10^2 + 6 \times 10 + 3.\end{aligned}$$

Hence, the square root of an arithmetical number may be obtained in practically the same manner as that of a polynomial.

The square root of 1 is 1; of 100 is 10; of 10000 is 100; etc. Hence the square root of a number between 1 and 100 is between 1 and 10; of a number between 100 and 10000 it is between 10 and 100; etc. That is, the integral part of the square root of an integer of *one* or *two* figures contains *one* figure; of an integer of *three* or *four* figures it contains *two* figures; etc. Therefore,

If the figures of an integer are marked off from right to left into groups of two, the number of figures in the integral part of the square root will be equal to the number of groups, any one figure that remains on the left being counted as a whole group.

Thus, marked off into groups, 20684 will become 2'06'84'. Since there are three groups, the integral part of the square root will contain three figures.

If the square root of a number be a number of two figures, the *tens* of the root may be denoted by a and the *ones* by b . Then the root will be expressed by $a + b$, and hence the number by $a^2 + 2ab + b^2$. The use of the fact that $\sqrt{a^2 + 2ab + b^2} = a + b$ is best shown by examples.

EXAMPLE 1.—Find the square root of 3969.

Pointing off, we have 39'69'. Since there are two groups, there must be two figures in the root. The root lies between 60 and 70, because 60^2 is less than 3969, and 70^2 is greater than 3969. That is, the *tens*' figure of the root is 6, *the square root of the largest square in the left-hand group of the given number.*

The work is indicated thus :

$$\begin{array}{r}
 \overline{a + b} \\
 39'69 \overline{60 + 3 = 63} \\
 a^2 = 36 \ 00 \\
 2a = 120 \overline{3 \ 69} \\
 2a + b = 123 \overline{3 \ 69}
 \end{array}$$

Since $a = 60$, $a^2 = 3600$. Subtracting, the remainder is 369. The trial divisor $2a$ becomes 2×60 , or 120. Dividing 369 by 120 gives approximately 3. Hence b is probably 3. Hence the trial divisor $2a + b$ becomes 123. Multiplying 123 by 3 gives 369. Taking this from the first remainder leaves zero. Hence, $60 + 3$, or 63, is the exact root.

EXAMPLE 2. — Find the square root of 203401.

By pointing off it is seen that there are three figures in the root.

$$\begin{array}{r}
 \overline{a + b + c} \\
 20'34'01' \overline{400 + 50 + 1 = 451} \\
 a^2 = 16 \ 00 \ 00 \\
 2a = 800 \overline{4 \ 34 \ 01} \\
 2a + b = 850 \overline{4 \ 25 \ 00} \\
 2(a+b) = 900 \overline{9 \ 01} \\
 2(a+b) + c = 901 \overline{9 \ 01}
 \end{array}$$

The largest square in 20 is 16. Hence $a^2 = 160000$, and $a = 400$. The first remainder is 43401. The first trial divisor $2a$ is 2×400 , or 800. Dividing 43401 by 800 gives approximately 50, the value of b . The first true divisor is $800 + 50$, or 850. Multiplying 850 by 50 gives 42500. Subtracting, the second remainder is 901. The second trial divisor $2(a+b)$ is 2×450 , or 900. 901 divided by 900 gives approximately 1, the value of c . The second true divisor is $900 + 1$, or 901. Multiplying 901 by 1 gives 901. Subtracting this from the second remainder leaves zero. Hence the exact root is $400 + 50 + 1$, or 451.

When the square root of a number has decimal places, the number itself has twice as many.

Thus, $0.23^2 = 0.0529$.

Hence, to mark off a number which contains a decimal, begin at the decimal point and mark to the left and to the right, putting two figures in each group.

If a decimal has an odd number of figures, add a zero to make it even.

Thus, 723.618 will become 7 23'.61'80'.

EXAMPLE 3. — Find the square root of 50.9796.

$$\begin{array}{r}
 50.'97'96' \overline{) 7.00 + .10 + .04 = 7.14} \\
 49. \quad 00 \quad 00 \\
 \hline
 14.00 \overline{) 1.9796} \\
 14.10 \overline{) 1.4100} \\
 \hline
 14.20 \overline{) .5696} \\
 14.24 \overline{) .5696}
 \end{array}$$

An approximation to the value of the square root of a number that is not a perfect square may be found to any degree of accuracy desired.

EXAMPLE 4. — Find to three decimal places the square root of 2. Since three decimal places are desired in the root, we annex 6 zeros.

$$\begin{array}{r}
 2.'00'00'00' \overline{) 1 + .4 + .01 + .004 = 1.414} \\
 1. \quad 00 \quad 00 \quad 00 \\
 \hline
 2 \overline{) 1.000000} \\
 2.4 \overline{) .960000} \\
 \hline
 2.8 \overline{) .040000} \\
 2.81 \overline{) .028100} \\
 2.82 \overline{) .011900} \\
 2.824 \overline{) .011296}
 \end{array}$$

EXERCISES

Find the square root of:

- | | | | |
|----------|-----------|-------------|-------------|
| 1. 7396. | 4. 26244. | 7. 16129. | 10. 146.41. |
| 2. 1849. | 5. 41209. | 8. 96.4324. | 11. 125.44. |
| 3. 5776. | 6. 17424. | 9. 12.25. | 12. 4.6225. |

Find to three decimal places the square root of:

- | | | | |
|--------|----------|-----------|-----------|
| 13. 5. | 15. 0.6. | 17. 28. | 19. 37.1. |
| 14. 3. | 16. 1.2. | 18. 0.45. | 20. 0.75. |

21. The area of a triangle whose sides are a , b , and c equals $\sqrt{s(s-a)(s-b)(s-c)}$, where s equals one half of the sum of the three sides. The sides of a triangular field are 42 rd., 38 rd., and 46 rd., respectively. Find its area to hundredths of a square rod.

NOTE. — The use of square root in solving equations will be shown in the next chapter.

137. Real and Imaginary Numbers. — Since any even power of either a positive or a negative number is positive, any *even root of a negative number* is neither a positive nor a negative number. The indicated even root of a negative number is called an **imaginary number**.

Thus, $\sqrt{-4}$, $\sqrt[4]{-9}$, $\sqrt[6]{-1}$, etc., are imaginary numbers.

A number that is not imaginary is called a **real number**. Hence, all numbers are classified as either real or imaginary. All numbers with which we have had to deal up to the present time are *real numbers*.

NOTE. — In the chapter following this some imaginary numbers will be encountered in solving equations. But no knowledge of the operations with these numbers will be needed in the *First Course*. A thorough treatment of imaginary numbers will be found in the *Second Course*.

138. Surds. — An indicated root which is real but the value of which cannot be computed exactly is called a **surd**.

Thus, $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt{m+n}$, etc., are surds.

A surd whose index is 2 is called a **quadratic surd**.

NOTE. — In this chapter are given only a few of the important processes applied to quadratic surds. A fuller treatment of surds will be found in the *Second Course*.

Every positive number has two square roots, differing only in sign. But in dealing with quadratic surds only the positive value of each will be considered.

139. Reducing a Surd to its Simplest Form. — A quadratic surd is in its **simplest form** when the expression under the radical sign is integral and contains no factor of which the indicated root can be found. A quadratic surd that is not in its simplest form can be reduced to its simplest form by use of the principle that

The square root of the product of two numbers equals the product of the square roots of the numbers.

Thus, $\sqrt{36} = \sqrt{4} \times \sqrt{9}$, because each equals 6.

$\sqrt{25 v^2 (v-w)^4} = \sqrt{25 v^2} \times \sqrt{(v-w)^4}$, because each equals $5 v (v-w)^2$.

The application of this principle in simplifying surds is shown in the following examples:

EXAMPLE 1. — Reduce to its simplest form $\sqrt{63}$.

Factoring 63 into two factors one of which is the largest perfect square possible, we have

$$\sqrt{63} = \sqrt{9 \times 7} = \sqrt{9} \times \sqrt{7} = 3\sqrt{7}.$$

EXAMPLE 2. — Reduce to its simplest form $\sqrt{\frac{4}{5}}$.

Multiplying both terms of $\frac{4}{5}$ by 5 to make the denominator a perfect square, then factoring, we get

$$\sqrt{\frac{4}{5}} = \sqrt{\frac{20}{25}} = \frac{\sqrt{20}}{\sqrt{25}} \times \sqrt{5} = \frac{2}{5} \sqrt{5}.$$

EXAMPLE 3. — Reduce to its simplest form $\sqrt{32 a^2 b^4 c^3}$.

$$\sqrt{32 a^2 b^4 c^3} = \sqrt{16 a^2 b^4 c^2 \times 2 c} = \sqrt{16 a^2 b^4 c^2} \times \sqrt{2 c} = 4 ab^2 c \sqrt{2 c}.$$

If the expression under the radical sign is integral, separate it into two factors of which one is the largest perfect square possible. The square root of the square factor becomes the coefficient of the radical.

If the expression under the radical sign is a fraction, multiply both terms of the fraction by such a quantity as will make the resulting denominator a perfect square. Then separate into two factors and proceed as above.

NOTE. — The importance of knowing how to simplify a quadratic surd is seen when the approximate value of a surd is to be found. Thus, if one knows that $\sqrt{2} = 1.414$, then $\sqrt{200} = 10\sqrt{2} = 14.14$. Also if it is known that $\sqrt{3} = 1.732$, then $\sqrt{27} = 3\sqrt{3} = 5.196$.

EXERCISES

Reduce to the simplest form :

- | | | | | |
|------------------|------------------|------------------|--------------------|--------------------|
| 1. $\sqrt{8}$. | 4. $\sqrt{27}$. | 7. $\sqrt{28}$. | 10. $\sqrt{54}$. | 13. $\sqrt{32}$. |
| 2. $\sqrt{12}$. | 5. $\sqrt{18}$. | 8. $\sqrt{48}$. | 11. $\sqrt{72}$. | 14. $\sqrt{80}$. |
| 3. $\sqrt{24}$. | 6. $\sqrt{50}$. | 9. $\sqrt{75}$. | 12. $\sqrt{125}$. | 15. $\sqrt{242}$. |

- | | | | | |
|-----------------------|----------------------------|------------------------------------|------------------------------|------------------------------|
| 16. $\sqrt{338}$. | 19. $\sqrt{\frac{1}{2}}$. | 22. $\sqrt{\frac{9}{20}}$. | 25. $\sqrt{\frac{25}{27}}$. | 28. $\sqrt{\frac{5}{21}}$. |
| 17. $\sqrt{432}$. | 20. $\sqrt{\frac{2}{3}}$. | 23. $\sqrt{\frac{16}{63}}$. | 26. $\sqrt{\frac{32}{40}}$. | 29. $\sqrt{\frac{8}{75}}$. |
| 18. $\sqrt{300}$. | 21. $\sqrt{\frac{4}{7}}$. | 24. $\sqrt{\frac{1}{24}}$. | 27. $\sqrt{\frac{7}{32}}$. | 30. $\sqrt{\frac{11}{96}}$. |
| 31. $\sqrt{a^2b}$. | 35. $\sqrt{8t^6v^3}$. | 39. $\sqrt{45AB^7}$. | | |
| 32. $\sqrt{4ab^4}$. | 36. $\sqrt{50P^4t}$. | 40. $\sqrt{a^3(a-b)^2}$. | | |
| 33. $\sqrt{25m^3n}$. | 37. $\sqrt{27W^3g^2}$. | 41. $\sqrt{\frac{a^2b^4}{cd^2}}$. | | |
| 34. $\sqrt{49x^5y}$. | 38. $\sqrt{5k^8v^5}$. | | | |

42. If $\sqrt{2} = 1.414$, find the values of the surds in Exercises 1, 5, 6, 11, 13, 15, 16, and 19.

43. If $\sqrt{3} = 1.732$, find the values of the surds in Exercises 2, 4, 8, and 9.

140. Addition and Subtraction of Surds.—Only quadratic surds that have the same number under the radical sign can be added or subtracted. They are added or subtracted just as any two similar terms.

Thus, just as $3n + 2n = 5n$, so $3\sqrt{7} + 2\sqrt{7} = 5\sqrt{7}$.

It usually is best first to reduce surds to be added or subtracted to their simplest forms before trying to unite them.

Thus, $\sqrt{12} + \sqrt{27} + \sqrt{75} = 2\sqrt{3} + 3\sqrt{3} + 5\sqrt{3} = 10\sqrt{3}$.

To add or subtract quadratic surds having the same radical part, add or subtract the coefficients of the radical part.

EXERCISES

Simplify :

- | | | |
|--|---|-------------------------------|
| 1. $\sqrt{2} + \sqrt{8}$. | 4. $\sqrt{45} - \sqrt{20}$. | 7. $2\sqrt{2} + \sqrt{32}$. |
| 2. $\sqrt{3} + \sqrt{12}$. | 5. $\sqrt{32} - \sqrt{8}$. | 8. $5\sqrt{5} - \sqrt{45}$. |
| 3. $\sqrt{18} + \sqrt{27}$. | 6. $\sqrt{50} - \sqrt{18}$. | 9. $2\sqrt{63} - \sqrt{28}$. |
| 10. $\sqrt{99} + \sqrt{44}$. | 12. $\sqrt{75} - \sqrt{3} - \sqrt{12}$. | |
| 11. $\sqrt{18} - \sqrt{8} + \sqrt{32}$. | 13. $\sqrt{20} + \sqrt{45} - \sqrt{80}$. | |

14. $\sqrt{6} + \sqrt{294} + \sqrt{24}.$

15. $2\sqrt{2} + 3\sqrt{18} - \sqrt{50}.$

16. $\sqrt{\frac{1}{2}} + 3\sqrt{\frac{2}{9}}.$

17. $3\sqrt{48} - \sqrt{\frac{1}{3}} - 2\sqrt{\frac{4}{3}}.$

18. $\sqrt{\frac{4}{5}} + 5\sqrt{\frac{1}{125}} - 2\sqrt{\frac{5}{4}}.$

19. $\sqrt{20} + 3\sqrt{5} + \sqrt{\frac{1}{5}}.$

20. $\sqrt{\frac{1}{8}} - 4\sqrt{3} - 6\sqrt{\frac{3}{4}}.$

21. $2a\sqrt{a^3} + 3\sqrt{a^5}.$

22. $\sqrt{x^3y^2} - 3y\sqrt{x^3}.$

23. $\sqrt{9a^4b^3} + 5ab\sqrt{a^2b}.$

24. $\sqrt{mn^3} + n\sqrt{9mn}.$

25. $8\sqrt{2v^3t} - v\sqrt{50v^3t}.$

26. $18p\sqrt{3pq^3} + 2q\sqrt{27p^3q}.$

27. $r\sqrt{t} + \sqrt{r^2t} + \sqrt{t^3}.$

28. $\sqrt{u^3vw} + \sqrt{uv^3w} + \sqrt{uvw^3}.$

29. $\sqrt{a^2bc} + \sqrt{a^2b^3c} + \sqrt{a^2bc^3}.$

141. Multiplication of Surds.—The product of two quadratic surds is obtained by use of the principle in § 139, *i.e.*

The product of the square roots of two numbers equals the square root of the product of the numbers.

Thus, $\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3}$ or $\sqrt{6}$; $\sqrt{15} \times \sqrt{5} = \sqrt{15 \times 5} = \sqrt{75} = 5\sqrt{3}.$

Similarly, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$; $\sqrt{m} \times \sqrt{mn} = \sqrt{m^2n} = m\sqrt{n}.$

The product of two expressions involving surds is obtained just as the product of any two polynomials, *i.e.* by multiplying each term of one by each term of the other, and adding the similar terms obtained.

EXAMPLE.—Multiply $3 + \sqrt{2}$ by $2 - \sqrt{2}.$

$$\begin{array}{r} 3 + \sqrt{2} \\ 2 - \sqrt{2} \\ \hline 6 + 2\sqrt{2} \\ - 3\sqrt{2} - 2 \\ \hline 6 - \sqrt{2} - 2 = 4 - \sqrt{2} \end{array}$$

EXERCISES

Multiply :

1. $\sqrt{3}$ by $\sqrt{5}.$

5. $\sqrt{2}$ by $\sqrt{6}.$

9. $\sqrt{2} \times \sqrt{14}.$

2. $\sqrt{2}$ by $\sqrt{7}.$

6. $\sqrt{3}$ by $\sqrt{15}.$

10. $\sqrt{5} \times \sqrt{35} \times \sqrt{21}.$

3. $\sqrt{3}$ by $\sqrt{11}.$

7. $\sqrt{21}$ by $\sqrt{3}.$

11. $\sqrt{3} \times \sqrt{6} \times \sqrt{2}.$

4. $\sqrt{5}$ by $\sqrt{13}.$

8. $\sqrt{10}$ by $\sqrt{30}.$

12. $\sqrt{8} \times \sqrt{22} \times \sqrt{33}.$

- | | |
|---|--|
| 13. $2 - \sqrt{5}$ by $4 + \sqrt{5}$. | 25. $2\sqrt{7} + 3\sqrt{2}$ by $3\sqrt{7} - 5\sqrt{2}$. |
| 14. $1 + \sqrt{3}$ by $3 + \sqrt{3}$. | 26. $1 - \sqrt{2}$ by $2\sqrt{3} + 3\sqrt{2}$. |
| 15. $7 - \sqrt{6}$ by $2 - \sqrt{6}$. | 27. $\sqrt{r}$ by $\sqrt{rs}$. |
| 16. $9 + \sqrt{11}$ by $3 - \sqrt{11}$. | 28. $a\sqrt{b}$ by $b\sqrt{a}$. |
| 17. $2 + \sqrt{3}$ by $3 + \sqrt{2}$. | 29. $2x\sqrt{xy}$ by $x^2\sqrt{x}$. |
| 18. $\sqrt{15} - 4$ by $\sqrt{5} + 1$. | 30. $p^2\sqrt{p}$ by $pq\sqrt{pq}$. |
| 19. $\sqrt{7} + 1$ by $3 - \sqrt{7}$. | 31. $5w\sqrt{wv}$ by $2v\sqrt{2w}$. |
| 20. $\sqrt{12} + 2$ by $\sqrt{3} - 5$. | 32. $\sqrt{3mn}$ by $\sqrt{7mn}$. |
| 21. $\sqrt{3} + \sqrt{2}$ by $\sqrt{3} - \sqrt{2}$. | 33. $x + \sqrt{y}$ by $x - \sqrt{y}$. |
| 22. $\sqrt{5} + \sqrt{6}$ by $\sqrt{5} + \sqrt{6}$. | 34. $r - \sqrt{s}$ by $r - \sqrt{s}$. |
| 23. $\sqrt{7} - 2\sqrt{2}$ by $\sqrt{7} - \sqrt{2}$. | 35. $2a - 3\sqrt{b}$ by $a + 2\sqrt{b}$. |
| 24. $\sqrt{3} + 4\sqrt{11}$ by $\sqrt{2} + \sqrt{11}$. | 36. $\sqrt{m} + \sqrt{n}$ by $\sqrt{m} - \sqrt{n}$. |

142. Division of Surds. — In the division of expressions involving surds it is best to first free the divisor of surds. This is accomplished by use of the principle that if the dividend and divisor are both multiplied by the same expression, the value of the quotient is unchanged.

EXAMPLE 1. — Divide $\sqrt{3}$ by $\sqrt{2}$.

Multiplying both dividend and divisor by $\sqrt{2}$,

$$\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3} \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{6}}{2} = \frac{1}{2}\sqrt{6}.$$

If the divisor is a binomial, in order to free it of surds it must be multiplied by the same binomial with the sign between the terms changed. This gives the product of the sum and difference of two numbers, which is the difference between their squares.

EXAMPLE 2. — Divide $\sqrt{7} + 2\sqrt{5}$ by $\sqrt{7} + \sqrt{5}$.

Multiplying both dividend and divisor by $\sqrt{7} - \sqrt{5}$,

$$\frac{\sqrt{7} + 2\sqrt{5}}{\sqrt{7} + \sqrt{5}} = \frac{(\sqrt{7} + 2\sqrt{5})(\sqrt{7} - \sqrt{5})}{(\sqrt{7} + \sqrt{5})(\sqrt{7} - \sqrt{5})} = \frac{\sqrt{35} - 3}{2} = \frac{1}{2}\sqrt{35} - 1\frac{1}{2}.$$

EXERCISES

Perform the indicated divisions:

- | | | | | |
|--------------------------------------|---|---|-------------------------------------|---------------------------------|
| 1. $\frac{1}{\sqrt{2}}$. | 4. $\frac{\sqrt{6}}{\sqrt{5}}$. | 7. $\frac{8}{\sqrt{6}}$. | 10. $\frac{a}{\sqrt{a}}$. | 13. $\frac{4}{\sqrt{2}p}$. |
| 2. $\frac{3}{\sqrt{3}}$. | 5. $\frac{\sqrt{6}}{\sqrt{2}}$. | 8. $\frac{12}{\sqrt{18}}$. | 11. $\frac{\sqrt{n}}{\sqrt{m}}$. | 14. $\frac{3v}{\sqrt{6}v}$. |
| 3. $\frac{10}{\sqrt{5}}$. | 6. $\frac{\sqrt{10}}{\sqrt{5}}$. | 9. $\frac{\sqrt{20}}{\sqrt{8}}$. | 12. $\frac{\sqrt{x+1}}{\sqrt{x}}$. | 15. $\frac{20r^2}{\sqrt{2}r}$. |
| 16. $\frac{2}{2+\sqrt{2}}$. | 21. $\frac{4}{\sqrt{21}-\sqrt{19}}$. | 26. $\frac{2\sqrt{5}-3\sqrt{3}}{3\sqrt{5}+\sqrt{3}}$. | | |
| 17. $\frac{4}{\sqrt{3}-1}$. | 22. $\frac{\sqrt{3}+1}{\sqrt{3}-1}$. | 27. $\frac{4\sqrt{7}+3\sqrt{2}}{3\sqrt{7}-7\sqrt{2}}$. | | |
| 18. $\frac{6}{\sqrt{3}+\sqrt{2}}$. | 23. $\frac{\sqrt{5}+2}{\sqrt{5}-2}$. | 28. $\frac{1}{a+\sqrt{b}}$. | | |
| 19. $\frac{16}{\sqrt{5}-\sqrt{3}}$. | 24. $\frac{\sqrt{11}-\sqrt{7}}{\sqrt{11}+\sqrt{7}}$. | 29. $\frac{x}{\sqrt{x}-\sqrt{y}}$. | | |
| 20. $\frac{12}{\sqrt{7}-\sqrt{6}}$. | 25. $\frac{\sqrt{6}+2\sqrt{2}}{\sqrt{6}+\sqrt{2}}$. | 30. $\frac{m-n}{\sqrt{m}-\sqrt{n}}$. | | |

143. Graphs of Quadratic Surds. — Although the exact value of a quadratic surd cannot be computed decimally, its value can be represented exactly by the length of a line. The principle involved is that the square of the hypotenuse of any right triangle equals the sum of the squares of the other two sides.

Thus, if a right triangle ABC be drawn with AB equal to 1 and BC equal to 1, then AC will equal $\sqrt{2}$. Now, if DC be drawn from C , perpendicular to CA , and equal to 1, the side DA of the right triangle ACD will equal $\sqrt{3}$.

By constructing another right triangle with AD as a side and the new side equal to 1, and continuing this process, the hypotenuse of a right triangle can be obtained that will represent any quadratic surd.

Many quadratic surds can be represented by drawing a single triangle. Thus, if $AB = 3$ and $BC = 1$, $AC = \sqrt{10}$. If $AB = 3$ and $BC = 2$, $AC = \sqrt{13}$.

EXERCISES

By drawing right triangles, represent the following:

- | | | | |
|-----------------|------------------|------------------|-------------------|
| 1. $\sqrt{5}$. | 4. $\sqrt{8}$. | 7. $\sqrt{34}$. | 10. $\sqrt{41}$. |
| 2. $\sqrt{6}$. | 5. $\sqrt{11}$. | 8. $\sqrt{53}$. | 11. $\sqrt{40}$. |
| 3. $\sqrt{7}$. | 6. $\sqrt{15}$. | 9. $\sqrt{45}$. | 12. $\sqrt{65}$. |

SUPPLEMENTARY EXERCISES

Find the square root of:

- $y^4 + 4y^3 + 4y^2 + 2y + 4 + \frac{1}{y^2}$.
- $R^2 + 8R + 12 - \frac{16}{R} + \frac{4}{R^2}$.
- $2 + \frac{a^2}{b^2} + 2ab + b^4 + \frac{2b^3}{a} + \frac{b^3}{a^2}$.
- $n^2 - tn - \frac{11t^2}{4} + \frac{3t^3}{2n} + \frac{9t^4}{4n^2}$.

Find the square root to hundredths of:

- $\frac{5}{7}$.

SUGGESTION. — In finding the square root of a common fraction it is best to first change the fraction to a decimal. Since we want two places of decimals in the root, write the first four decimals of the number. Thus, $\frac{5}{7} = 0.7142$.

- | | | | | |
|--------------------|--------------------|----------------------|--------------------|---------------------|
| 6. $\frac{1}{2}$. | 7. $\frac{6}{7}$. | 8. $\frac{11}{15}$. | 9. $\frac{3}{4}$. | 10. $\frac{7}{8}$. |
|--------------------|--------------------|----------------------|--------------------|---------------------|

11. A method used by the Arabs, and as late as the Middle Ages, of finding the approximate value of the square root of a number was to substitute in the formulæ of approximation:

$$(a) \sqrt{n^2 + a} = n + \frac{a}{2n}, \quad (b) \sqrt{n^2 + a} = n + \frac{a}{2n + 1}.$$

The true value of the root was found to be between the two values obtained. For example, to find $\sqrt{82}$, separate 82 into 81 + 1. Then from

$$(a) \sqrt{82} = \sqrt{81} + 1 = \sqrt{9^2 + 1} = 9 + \frac{1}{18} = 9\frac{1}{18}; \text{ from}$$

$$(b) \sqrt{82} = \sqrt{81 + 1} = \sqrt{9^2 + 1} = 9 + \frac{1}{19} = 9\frac{1}{19}.$$

Find the decimal values of these results, which are the same to the hundredths figure, and hence find the true square root to hundredths.

Find by the method of Problem 11 between what values the square roots of the following lie:

12. 17.

13. 50.

14. 65.

15. 102.

16. 147.

Reduce to the simplest form:

17. $\sqrt{\frac{1}{x^2} + \frac{2}{x} + 3}.$

18. $\sqrt{\frac{m}{n^2} + \frac{n}{m^2}}.$

19. $\sqrt{\frac{B^2}{4A^2} - \frac{C}{A}}.$

20. Divide $1 + \sqrt{2}$ by $1 + \sqrt{2} + \sqrt{3}$.

SUGGESTION. — To free the divisor of surds it will be necessary to multiply both dividend and divisor twice when the divisor is a trinomial such as this. First multiply by the expression obtained by changing one of the signs in the divisor. Thus,

$$\frac{1 + \sqrt{2}}{1 + \sqrt{2} + \sqrt{3}} = \frac{(1 + \sqrt{2})(1 + \sqrt{2} - \sqrt{3})}{(1 + \sqrt{2} + \sqrt{3})(1 + \sqrt{2} - \sqrt{3})} = \frac{3 + \sqrt{2} - \sqrt{3} - \sqrt{6}}{2\sqrt{2}} = \text{etc.}$$

Divide:

21. $\sqrt{3} + \sqrt{5}$ by $1 + \sqrt{3} + \sqrt{5}$.

25. $\sqrt{5} - \sqrt{3}$ by $\sqrt{5} - \sqrt{3} - 1$.

22. $1 + \sqrt{2}$ by $1 - \sqrt{2} + \sqrt{5}$.

26. $\sqrt{x}$ by $\sqrt{x} + \sqrt{y} + \sqrt{z}$.

23. 2 by $\sqrt{2} + \sqrt{3} + \sqrt{7}$.

27. $\sqrt{a} - \sqrt{b}$ by $\sqrt{a} + \sqrt{b} - \sqrt{c}$.

24. $\sqrt{2}$ by $\sqrt{2} + \sqrt{3} + \sqrt{5}$.

28. $1 - \sqrt{m}$ by $1 + \sqrt{m} + \sqrt{n}$.

CHAPTER XIV

QUADRATIC EQUATIONS

144. Quadratic Equations. — As shown in § 73 and § 96, some problems may be expressed and solved by use of equations of the **second degree**, or **quadratic equations** (see § 57 for definitions). The equations discussed in § 73 and § 96 were of very special types, and the methods of solution there shown are not applicable to all quadratic equations. We shall now discover how to solve any quadratic equation whatever.

A quadratic equation that contains a term of the first degree in the unknown number is called a **complete quadratic equation**. One that does not contain a term of the first degree in the unknown number is called a **pure quadratic equation**.

Thus, $3n^2 - 4n = 5$ is a complete quadratic equation. And $7t^2 - 9 = 0$ is a pure quadratic equation.

By simplifying and combining similar terms, if necessary, any quadratic equation whose unknown number is denoted by x can be written in the form $Ax^2 + Bx + C = 0$, where A , B , and C are known numbers.

145. Quadratics solved by Factoring. — Some quadratics can be solved by *factoring*, as shown in § 96. The rule there given should now be reviewed and applied in the following exercises.

NOTE. — It must be remembered that every quadratic equation has *two roots*. *Both roots* should be checked by seeing if they satisfy the given equation.

EXERCISES

Solve by factoring, and check:

1. $a^2 = 7a - 12$.
2. $P^2 + 3P = 10$.
3. $w^2 + 9w + 14 = 0$.
4. $2x^2 + 5x = 3$.
5. $6V^2 + 13V + 6 = 0$.
6. $10t^2 + 33t = 7$.
7. $6y^2 = 23y + 4$.
8. $2 - R = 21R^2$.
9. $54z = 9z^2 + 72$.
10. $4A = A^2 - 77$.
11. $\frac{m}{2} + \frac{1}{2} = \frac{1}{m}$.
12. $1 + \frac{28}{D^2} = \frac{11}{D}$.
13. $15K + 4 = \frac{3}{K}$.
14. $5 = \frac{4}{r} + \frac{33}{r^2}$.
15. $a^2 + \frac{1}{8}a = 3\frac{3}{4}$.
16. $1 + \frac{22}{P} - 6P = 0$.
17. $6 - \frac{50}{s^2} + \frac{55}{s} = 0$.
18. $5r - 8 = \frac{21}{r}$.
19. $\frac{3z^2 - 4}{z + 2} = 2z + 1$.
20. $\frac{y + 15}{y + 3} = y + 8$.
21. $\frac{3b - 10}{14} = \frac{b + 120}{b}$.
22. $\frac{1}{2x - 2} - \frac{x - 7}{x + 1} + \frac{x + 24}{5x^2 - 5} = 0$.
23. $\frac{N}{N - 3} - \frac{N - 1}{N + 3} = 2$.
24. $\frac{s + 3}{2} = \frac{10}{3} - \frac{2}{s + 3}$.
25. $\frac{3(t - 1)}{t + 1} = 5 + \frac{2(t + 1)}{t - 1}$.
26. $\frac{r^2 + r + 1}{r - 1} - 6 = \frac{r^2 - r + 1}{r + 1}$.
27. $3a = a^2 - 18$.

146. Quadratics solved by Square Root.—Any pure quadratic equation can be solved by the method of § 73.

To solve a pure quadratic equation first, by clearing of fractions, transposing terms, etc., reduce the equation to the form $x^2 = A$, where x is the unknown number and A is known. Take the square root of both members of the resulting equation, attaching the double sign $\pm$ to the second member.

EXAMPLE 1.—Solve $8t^2 = 2t(t - 5) + 10(t + 2)$.

Removing parentheses, transposing, combining terms, and dividing by the coefficient of t ,

$$t^2 = \frac{10}{3}.$$

Taking square root,

$$t = \pm \sqrt{\frac{10}{3}}$$

$$= \pm \frac{1}{3} \sqrt{30}.$$

These roots may be expressed decimally, if desired. Thus, $\sqrt{30} = 5.477$, approximately. Hence, $t = \pm 1.825$, approximately.

NOTE.—In some problems in which the roots involve surds it is sufficient to express the roots with radical signs. But in many applied problems it is desirable to compute the approximate values of such roots decimally.

The student should be able to compute the roots in either form when desirable.

EXAMPLE 2. — Solve $D^2 + 5 = 0$.

Transposing,

$$D^2 = -5.$$

Taking square root,

$$D = \pm \sqrt{-5}, \text{ imaginary numbers.}$$

EXERCISES

Solve:

1. $7y^2 - 28 = 0$.
2. $10V^2 - 150 = 4V^2$.
3. $r^2 = 845 - 4r^2$.
4. $a(a+4) = 4a + 49$.
5. $(M+2)(2M+3) = 7(M+3)$.
6. $8t^2 + 72 = 0$.
7. $5n^2 + 20 = 0$.
8. $4A^2 = 7$.
9. $6R^2 - 8 = 0$.
10. $5d^2 + 10 = 0$.
11. $20P^2 + 8 = 0$.
12. $\frac{x}{125} = \frac{5}{x}$.
13. $\frac{3t}{4} = \frac{4}{3t}$.
14. $\frac{1}{y+1} = \frac{y-1}{6}$.
15. $\frac{a^2+4}{8} = 1$.
16. $\frac{V^2+8}{2} = 2$.
17. $\frac{9s^2-1}{3} = \frac{3}{4}$.
18. $\frac{5}{v^2-1} = 4$.
19. $\frac{3}{2k^2} - \frac{1}{k^2} = \frac{9}{4}$.
20. $\frac{1-b^2}{6} - \frac{b^2-1}{5} = 1$.
21. $\frac{n-1}{n+1} + \frac{n+1}{n-1} = 3$.

22. $\frac{3A^2-7}{A^2-1} - \frac{A-1}{A+1} = \frac{A+1}{A-1}$. 26. $\frac{3s+2}{2+s} - \frac{13}{2s+3} = 0$.
23. $\frac{Q+5}{6} - \frac{8}{Q+1} = 1$. 27. $2T+1 + \frac{(3T+1)(T-2)}{T+2} = 0$.
24. $\frac{z+6}{6z+1} - \frac{6z+1}{z+6} = 0$. 28. $\frac{2G}{5} - \frac{1+G^2}{2G+3} = \frac{3}{5}$.
25. $\frac{4p+1}{3p-4} = \frac{8p-19}{7p-24}$. 29. $\frac{x^2+x+1}{x-1} - 6 = \frac{x^2-x+1}{x+1}$.

Find approximate to hundredths the roots of:

30. $x^2 = 7$. 36. $5B^2 - 14 = 0$.
31. $4a^2 - 10 = 0$. 37. $3t^2 = 28$.
32. $8t^2 - 12 = 0$. 38. $(p+4)(p-4) = 4$.
33. $15n^2 - 4 = 0$. 39. $\frac{2}{r+3} = \frac{r-3}{4}$.
34. $7V^2 = 26$. 40. $\frac{K+5}{4} - \frac{3}{K-1} = 1$.
35. $2D^2 = 17$.

147. Quadratics solved by completing the Square. — As shown in § 86, any trinomial that is a perfect square may be written in either the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$. Hence, if we have given either the binomial $a^2 + 2ab$ or $a^2 - 2ab$, to make it a perfect square we must add b^2 , *i.e. the square of one half of the coefficient of a in the term $2ab$* . This process is called **completing the square**.

Thus, to complete the square whose first two terms are $n^2 - 10n$, we add the square of $\frac{1}{2}$ of 10, or 25. This gives $n^2 - 10n + 25$, which is the square of $n - 5$.

By use of this process of completing the square any complete quadratic equation whatever may be solved, as shown in the following examples.

EXAMPLE 1. — Solve $v^2 + 3v - 10 = 0$.

Transposing, $v^2 + 3v = 10$.

Adding the square of $\frac{1}{2}$ of 3 to both members, to complete the square on the left,

$$v^2 + 3v + \frac{9}{4} = 10 + \frac{9}{4}.$$

Taking square root,

$$v + \frac{3}{2} = \pm \frac{7}{2}.$$

Transposing,

$$v = -\frac{3}{2} \pm \frac{7}{2}, \text{ i.e. } -\frac{3}{2} + \frac{7}{2} \text{ or } -\frac{3}{2} - \frac{7}{2}.$$

Simplifying,

$$v = 2 \text{ or } -5$$

EXAMPLE 2. — Solve $5P^2 + 12P = 2$.

Dividing by 5,

$$P^2 + \frac{12}{5}P = \frac{2}{5}.$$

Adding square of $\frac{1}{2}$ of $\frac{12}{5}$, $P^2 + \frac{12}{5}P + \frac{36}{25} = \frac{2}{5} + \frac{36}{25}$.

Taking square root,

$$P + \frac{6}{5} = \pm \frac{1}{5}\sqrt{46}.$$

Transposing,

$$P = -\frac{6}{5} \pm \frac{1}{5}\sqrt{46}.$$

If the approximate values of these roots, expressed decimally, are desired, they may be found by taking the square root of 46 and simplifying. Thus, $\sqrt{46} = 6.782$, approximately. Hence, $P = 0.15$ or -2.55 , approximately.

EXAMPLE 3. — Solve $y^2 = 4y - 7$.

Transposing,

$$y^2 - 4y = -7.$$

Adding square of $\frac{1}{2}$ of 4,

$$y^2 - 4y + 4 = -3.$$

Taking square root,

$$y - 2 = \pm \sqrt{-3}.$$

Transposing,

$$y = 2 \pm \sqrt{-3}.$$

These roots involve *imaginary numbers*.

Evidently, any complete quadratic equation may be solved by the following rule:

(1) Reduce the equation to the form $x^2 + px = q$, where x is the unknown number.

(2) Add to each member the square of one half of the coefficient of the term in the first power of the unknown number.

(3) Take the square root of both members, attaching the double sign $\pm$ to the second member.

(4) Solve the resulting linear equations.

EXERCISES

Solve by completing the square:

1. $y^2 - 6y + 8 = 0.$

6. $x^2 + x = 6.$

2. $n^2 + 4n = 12.$

7. $V^2 - 11V + 30 = 0.$

3. $A^2 - 8A = 9.$

8. $d^2 = 14 - 5d.$

4. $t^2 = 22 + 9t.$

9. $M^2 + 12M + 15 = 0.$

5. $v^2 + 14v + 40 = 0.$

10. $2k^2 + k = 4.$

11. $2P^2 - 5P = 3.$

12. $6r^2 + 6 = 13r.$

13. $4W^2 - 11W = 3.$

14. $10n^2 = 29n - 10.$

15. $3R^2 = 17R + 28.$

16. $2b^2 + 3b = 4.$

17. $5k^2 = k + 7.$

18. $12G^2 + 3 = 14G.$

19. $9E^2 = 6E + 26.$

20. $16a^2 - 96a = 1792.$

21. $N^2 - 7N + 15 = 0.$

22. $\frac{1}{t+1} - \frac{2}{1-t} = \frac{13}{4t-1}.$

23. $\frac{5}{4+r'} + \frac{5}{4-r'} = \frac{8}{3}.$

24. $\frac{S^2}{2} - \frac{S}{3} + 20\frac{1}{2} = 42\frac{2}{3}.$

25. $6H + \frac{35-3H}{H} = 42.$

26. $x^2 + 7x = 7(x+3) + 4.$

27. $\frac{w-3}{w+5} - \frac{w+4}{w-7} = 2.$

28. $\frac{3}{p^2-1} + \frac{1}{p+1} = 1 + \frac{2}{p-1}.$

29. $\frac{1}{2V-3} + \frac{2}{V+1} + 3 = 0.$

30. $3 - \frac{1}{y+2} = \frac{1}{y-2}.$

31. $\frac{1}{U-1} + \frac{1}{U-2} + \frac{1}{U-3} = 0$

32. $\frac{c+2}{c-1} - \frac{4-c}{2c} = 2\frac{1}{3}.$

33. $(z-5)^2 + (z-10)^2 = 37.$

34. $\frac{r_1+3}{r_1+9} + \frac{r_1-4}{r_1-1} = 0.$

35. $\frac{3}{B^2-1} + \frac{1}{2B-2} = \frac{1}{4}.$

36. $\frac{1}{m+2} - \frac{1}{m-2} = \frac{1}{1-m}.$

37. $\frac{2Y-1}{Y-1} + \frac{Y-1}{Y-2} = \frac{3Y-10}{Y-3}.$

38. $\frac{d-4}{2d-12} - \frac{d-4}{3d-16} = \frac{11}{2}.$

Find approximate to hundredths the roots of:

39. $n^2 = 3n + 7.$

40. $u^2 - 15u = 60.$

41. $9A^2 = 1 - 3A.$

42. $D^2 + 8D + 11 = 0.$

43. $z^2 + 6z = 1.$

44. $12R^2 = R + 2.$

45. $34t = 3t^2 - 105.$

46. $19 = a + 4a^2.$

47. $5M^2 = 2M + 7.$

48. $21 - 2v = 4v^2.$

148. Quadratics solved by a Formula.—Since any quadratic equation may be written in the general form,

$$Ax^2 + Bx + C = 0,$$

where x denotes the unknown number and A , B , and C are known numbers, the roots of this equation will give a *formula* by which the roots of any particular quadratic equation may be written down at once.

$Ax^2 + Bx + C = 0$ is solved by completing the square as follows:

$$\text{Dividing by } A, \quad x^2 + \frac{B}{A}x + \frac{C}{A} = 0.$$

$$\text{Transposing,} \quad x^2 + \frac{B}{A}x = -\frac{C}{A}.$$

$$\text{Completing square,} \quad x^2 + \frac{B}{A}x + \frac{B^2}{4A^2} = \frac{B^2 - 4AC}{4A^2}.$$

$$\text{Taking square root,} \quad x + \frac{B}{2A} = \pm \frac{\sqrt{B^2 - 4AC}}{2A}.$$

$$\begin{aligned} \text{Transposing,} \quad x &= -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A} \\ &= \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \end{aligned}$$

Now, by replacing A , B , and C in the formula for the roots

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

by the particular values which they have in any given equation, the roots of any quadratic equation may be written down at once. Thus the long process of completing the square in every equation may be avoided. Hence, *this formula should be mastered and used in all future work where the equation cannot be readily solved by factoring.*

EXAMPLE 1. — Solve $3n^2 - 4n = 15$.

Written in the form $Ax^2 + Bx + C = 0$, this becomes

$$3n^2 - 4n - 15 = 0.$$

Here $A = 3$, $B = -4$, and $C = -15$. Substituting these values for A , B , and C in the formula,

$$\begin{aligned} n &= \frac{4 \pm \sqrt{16 + 180}}{6} \\ &= \frac{4 \pm 14}{6} \\ &= 3 \text{ or } -\frac{5}{3}. \end{aligned}$$

EXAMPLE 2. — Solve $4V^2 + 2 = -3V$.

Transposing, $4V^2 + 3V + 2 = 0$.

Here $A = 4$, $B = 3$, and $C = 2$.

Substituting,
$$V = \frac{-3 \pm \sqrt{9 - 32}}{8}$$

$$= \frac{-3 \pm \sqrt{-23}}{8}, \text{ imaginary expressions}$$

EXERCISES

Solve by use of the formula.

1. $y^2 - 3y = 10$.
2. $p^2 + 10p + 21 = 0$.
3. $2A^2 + A = 6$.
4. $3t^2 + 5t = 2$.
5. $6K^2 + K - 5 = 0$.
6. $7u^2 = 50u - 7$.
7. $4M^2 + M + 5 = 0$.
8. $6r_1^2 + 6 = 13r_1$.
9. $10a^2 + 14 = 39a$.
10. $8D^2 = 65D - 8$.
11. $2n^2 + 3n + 4 = 0$.
12. $5z^2 = 2z - 7$.
13. $15q^2 + 3 = q$.
14. $B^2 + 4 = 8B$.
15. $2v^2 = 7v - 6$.
16. $4k^2 + 5 = 6k$.
17. $21c^2 = c + 4$.
18. $9y^2 = 6 + y$.
19. $12 + x = 11x^2$.
20. $1 - a^2 - 3a = 0$.
21. $\frac{2}{n} = \frac{n+1}{2}$.
22. $\frac{3}{1-t} - \frac{t-1}{3} = 0$.
23. $\frac{4}{v^2} = 2 + \frac{1}{2v}$.
24. $\frac{1}{k} + k = 1\frac{1}{2}$.
25. $R + 3 = \frac{4}{R-1}$.
26. $s - \frac{1}{s} = 3 - \frac{1}{3}$.
27. $\frac{2a+3}{a-2} = \frac{a+3}{2a-1}$.
28. $\frac{E-1}{E-4} + \frac{E+9}{E+3} = 0$.
29. $x + 2 = \frac{3x^2 - 4}{2x + 1}$.
30. $1 + V = \frac{4V}{1 + V}$.
31. $b + \frac{1}{b} = 1$.

$$32. 1 - m = \frac{1 + 2m^2}{5 + m}.$$

$$34. \frac{G-1}{G+1} = \frac{G+1}{G-1} - 1.$$

$$33. \frac{6}{A+1} + \frac{2}{A} = 3.$$

$$35. \frac{3}{x-4} = \frac{6}{x-5} - \frac{8}{x-3}.$$

$$36. \frac{A-2}{A+2} + \frac{A+2}{A-2} - \frac{10}{4-A^2} = 0.$$

$$37. \frac{2}{5w-10} - \frac{1-w}{w+2} = \frac{w+1}{w^2-4}.$$

$$38. \frac{7-3M}{4-3M} + \frac{7-M}{2M+2} = \frac{2M+3}{4M-2}.$$

$$39. \frac{1}{18} + \frac{y}{y+1} = \frac{y+3}{2y+8}.$$

$$40. \frac{1}{8} + \frac{1}{4x+4} = \frac{3}{2x^2-2}.$$

$$43. 4W - \frac{14-W}{W+1} = 14.$$

$$41. \frac{8+P}{8-P} + \frac{4-2P}{4+2P} = \frac{8}{3}.$$

$$44. \frac{(n-1)^2}{2} + \frac{n-1}{4} = 9.$$

$$42. 5s - \frac{3s-3}{s-3} = 2s + \frac{3s-6}{2}.$$

$$45. \frac{3+T}{4+T} - \frac{T-5}{T-6} = \frac{1}{12}.$$

Find to hundredths the roots of:

$$46. a^2 - 5a = 8.$$

$$48. 12d^2 = 3d + 19.$$

$$50. 6Q^2 + 1 = 9Q.$$

$$47. 4N^2 + N - 7 = 0.$$

$$49. 9 - 2y = y^2.$$

$$51. 18 = 3B + 2B^2.$$

149. Problems solved by Quadratic Equations.—It has been seen that the roots of quadratic equations may be fractions, negative numbers, surds, or even imaginary numbers. In solving a problem by means of a quadratic equation, it is necessary to consider the nature of the roots to see if both roots satisfy all requirements of the given problem. Thus, a problem may require for its solution a *positive* number. If one of the roots of the equation formed is found to be *negative*, it must be discarded, although it satisfies the equation. If the nature of a problem requires a *real* number for its solution, and the roots of the equation formed are found to be *imaginary*, they must be discarded.

EXAMPLE 1. — A train runs at a uniform rate between two points 280 miles apart. If it ran 5 miles per hour faster, it would make the distance in 1 hour less time. Find the rate of the train.

Let v equal the rate of the train.

Then $\frac{280}{v}$ = number of hours required for the run.

And $\frac{280}{v+5}$ = number of hours required at the increased speed.

Hence,
$$\frac{280}{v} = \frac{280}{v+5} + 1.$$

Solving,
$$v = 35 \text{ or } -40.$$

But the rate must be a positive number. Hence, the root -40 must be discarded, and the only rate possible is 35 miles per hour.

EXAMPLE 2. — Find the real number whose square increased by 32 equals 8 times the number.

Let n = the number.

Then,
$$n^2 + 32 = 8n.$$

Solving,
$$n = 4 \pm \sqrt{-16}.$$

Since both roots involve imaginary numbers, the problem has no solution, *i.e.* it is *impossible*.

EXERCISES

By use of one unknown number solve:

1. Find two consecutive arithmetical numbers the sum of whose squares is 313.

2. Find two consecutive arithmetical numbers whose product is 240.

3. Divide 31 into two parts the sum of whose squares is 541.

4. Find two arithmetical numbers whose sum is 50 and product 336.

5. Divide 14 into two parts such that 4 times the square of the larger shall exceed 6 times the square of the smaller by 40.

6. The denominator of a certain fraction is 5 more than the numerator. If the fraction be added to the fraction inverted, the sum will be $2\frac{5}{4}$. Find the fraction.

7. A pupil was to divide 12 by a certain number, but by mistake he subtracted the number from 12. His result was 5 too great. Find the number.

8. The side of one square exceeds that of another by 3 inches, and its area exceeds twice the area of the other by 17 square inches. Find the lengths of their sides.

9. A rectangular field is 96 ft. longer than it is wide, and it contains 298,000 sq. ft. What are its dimensions?

10. A floor can be paved with 200 square tiles of a certain size. If each tile were 1 in. shorter each way, it would require 288 tiles. Find the size of each tile.

11. In the center of a rectangular room is a rug 9 ft. by 12 ft. Around this is a border of uniform width. The area of the floor is 208 sq. ft. Find the width of the border.

12. A lawn is 40 ft. by 90 ft. Two boys agree to mow it. The first boy is to mow one half of it by cutting a strip of uniform width around it. How wide a strip must he cut?

13. A farmer has a field of wheat 80 rd. wide and 120 rd. long. How wide a strip must he cut around the field in order to have one fourth of the wheat cut?

14. In Problem 13, how wide a strip must he cut around the field in order to have one half of the wheat cut?

15. A farmer has a field 40 rd. wide and 80 rd. long to be plowed and planted in corn. He wishes to plow $4\frac{1}{2}$ acres the first day by plowing a strip of uniform width around it. How wide a strip must he plow?

16. The hypotenuse of a right triangle is 4 in. longer than one leg and 2 in. longer than the other. Find the sides of the triangle.

17. One side of a rectangle is 4 in. longer than the other, and its diagonal is 20 in. How long are the sides?

18. Two men start at the same time from the intersection of two roads, one driving south at the rate of 4 miles an hour and

the other west at the rate of 3 miles an hour. In how many hours will they be 25 miles apart?

19. Two trains are 100 miles apart on perpendicular roads, and are approaching the crossing. One train runs 10 miles an hour faster than the other. At what rates must they run if they both reach the crossing in 2 hours?

20. A train running 6 miles an hour slower than usual, due to a storm, was 1 hour late in making a run of 252 miles. Find its speed.

21. An engineer increased the speed of his train 5 miles an hour, and made a run of 360 miles in 1 hour less than schedule time. What was the speed when running according to schedule?

22. A company of people arranged for a dinner, for which they contracted to pay \$45. Five of the people were unable to attend, and as a result each of the others had to pay 30 cents more than he had expected. How many were present?

23. A farmer sold his wheat for \$540. A month later the price of wheat advanced $10\frac{1}{2}$ ¢ a bushel, and he found that if he had held his wheat until that time, he could have received the same money and have had 75 bu. left. How much per bushel did he receive for the wheat?

24. A trader bought a flock of sheep for \$1020. Two of them died, and he sold the rest at a profit of \$3 a head. He made \$225 on the transaction. How many did he buy?

25. A man bought a farm for \$20,000. Later he sold all except 35 acres of it at a gain of \$35 per acre over the cost, and received just what he paid for the whole farm. How many acres in the farm?

26. One pump can fill a tank in 6 minutes less time than is required for another pump to fill it. The two working together can fill it in $10\frac{2}{3}$ minutes. Find the time required for each pump alone to fill the tank.

27. One of two pumps can fill a tank in 28 minutes, and the time required for the other pump to do it is $19\frac{4}{9}$ minutes longer than is required for the two pumps together to fill it. Find the time required for the two pumps together to fill the tank.

28. With A's help B could do a piece of work in $6\frac{1}{4}$ hours less time than would be required for B to do it alone. A could do it alone in 21 hours. In how many hours could B do it alone?

29. One pipe could empty a tank in 2 hours less time than would be required for a second pipe to empty it. After the first pipe has been open 4 hours, it is closed and the second pipe is opened. The second pipe finishes emptying the tank in 5 hours more. In what time could each pipe alone empty it?

30. An aviator flew 80 miles against the wind, and back again, in 6 hours. His speed against the wind was 10 miles per hour greater than the speed of the wind. Find the speed of the wind, also what his speed would have been had there been no wind.

31. A line 16 in. long is to be divided into two parts such that the ratio of the whole line to the longer part shall be equal to the ratio of the longer part to the shorter part. Find to tenths of an inch the lengths of the parts.

NOTE.—The line in Problem 31 is said to be divided in “extreme and mean ratio.” Since the time of the Greeks the problem to divide a line in extreme and mean ratio has been called the “Problem of the Golden Section.”

32. Experience has shown that a book, photograph, front of a tall building, or other rectangular object, is most pleasing to the eye when its length and width are obtained by dividing their sum in extreme and mean ratio.

The page of a book is to be 5 in. wide. How long should it be made to be most pleasing to the eye?

SUGGESTION.—If n denotes the length, $\frac{n+5}{n} = \frac{n}{5}$.

33. A photograph is to be mounted on cardboard 6 in. wide. How long should the board be cut?

34. An architect is designing a building whose frontage on the street is to be 80 ft. How high should he make it for the greatest beauty?

35. An open box that shall hold 384 cu. in. is to be made from a square piece of cardboard by cutting out a 6-inch square from each corner and turning up the sides. Find the size of the piece of cardboard that must be used.

36. An open box that shall contain 324 cu. in. and that shall be twice as long as wide is to be made from a rectangular piece of tin by cutting out 2-inch squares from the corners and bending up the sides. Find the dimensions of the piece of tin required.

37. A flat disk of sheet metal 3 inches in radius is to be stamped by means of a die into a box lid 1 inch deep. The total area of the lid must equal the area of the flat disk. What will be the radius of the lid?

SUGGESTION. — If r is the radius of the lid, its total surface is composed of a circle whose area is πr^2 and a cylindrical part whose area is $2\pi r$. Hence, $\pi r^2 + 2\pi r = 9\pi$.

38. If the radius of the disk in Problem 37 is 2 inches and the depth of the lid $\frac{1}{2}$ inch, what will be the radius of the lid?

39. The rim of the front wheel of a carriage is 2 ft. less than the rim of the hind wheel. The front wheel makes 88 more revolutions in going a mile than the hind wheel. Find the length of the rim of each wheel.

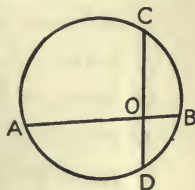
40. The circumference of the little wheel on a sewing machine is 29 inches less than that of the big wheel which drives it. While the belt moves a distance of 1275 inches, the little wheel makes 116 more revolutions than the big one. Find the circumference of each wheel.

41. A park is 900 ft. wide and 1650 ft. long. It is desired to move out the boundary lines the same distance along one side and both ends so that the area inclosed will be just twice as great as it now is. Find how far the boundary lines must be moved.

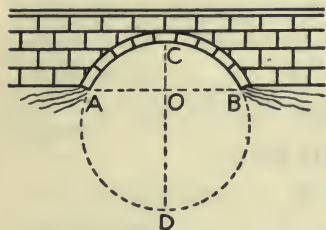
42. The longer base of a trapezoid is 14 in. longer than the other base, the altitude is equal to the shorter base, and the area is 120 sq. in. Find the lengths of the bases and the altitude.

43. If any two chords AB and CD of a circle meet at O , then $AO \times OB = CO \times OD$.

If $CO = 3$ in., $OD = 2$ in., and $AB = 8$ in., find AO and OB .



44. In Problem 43, if $CO = 9$, $OD = 7$, and $AB = 22$, find AO and OB .



45. AB , the *span* of the stone arch ACB that is to be constructed, is 40 ft. ($AO = 20$ ft. and $OB = 20$ ft.), and the diameter CD of the arch is 50 ft. Find CO , the *rise* of the arch.

46. If the span of an arch is 16 ft. and the diameter 20 ft., find the rise of the arch

47. If the span of an arch is 20 ft. and the diameter 30 ft., compute the rise.

150. Literal Equations: Formulæ. — If a literal equation or formula (see § 113) is of the second degree with respect to the unknown number, it may be solved by the methods of this chapter.

EXAMPLE. — Solve $t(t - b) = a(a + b)$ for t .

Removing parentheses, $t^2 - bt = a^2 + ab$.

Transposing, $t^2 - bt - (a^2 + ab) = 0$.

In formula, $A = 1$, $B = -b$, $C = -(a^2 + ab)$.

Substituting in formula, $t = \frac{b \pm \sqrt{b^2 + 4(a^2 + ab)}}{2}$.

$$= \frac{b \pm (2a + b)}{2}$$

$$= a + b \text{ or } -a.$$

EXERCISES

1. Solve $an^2 = b$ for n .
2. Solve $a^2v^2 - a^4 = 0$ for v .
3. Solve $mn = m^2 - 4$ for m .
4. Solve $2x^2 - 3r^2 = 5xr$ for x ; for r .
5. Solve $W^2 + 2A^2 = 3WA$ for W ; for A .
6. Solve $y^2 + 2my = n^2 + 2mn$ for y ; for n .
7. Solve $w^2 + 2rw + s = 0$ for w .
8. Solve $z + \frac{1}{n} = n + \frac{1}{z}$ for n ; for z .
9. Solve $P^2 + 2P = vP + 2v$ for P .
10. Solve $t^2 + at + bt + ab = 0$ for t .
11. Solve $A^2 - (m - n)A = mn$ for A .
12. Solve $\frac{1 + R}{1 - Rn} + \frac{1 - R}{1 + Rn} = 1$ for n ; for R .
13. Solve $S = \frac{1}{2}n[2a + (n - 1)d]$ for n .
14. Solve $S = \frac{n(n - 1)}{2}$ for n .

15. If an object is let fall downward from a height s feet above the earth, the time in seconds required for it to strike the earth is computed by the formula $s = 16t^2$. Solve for t .

16. How long would it take a body to fall a distance of 1280 feet?

17. If an object is thrown downward toward the earth with an initial velocity of v feet per second, the distance s in feet that it will fall in t seconds is computed by the formula $s = vt + 16t^2$. Solve for t .

18. An object is thrown downward with a velocity of 32 feet a second from a distance of 240 feet above the earth. How long will it take for it to strike the ground?

19. The velocity of the discharge of water from a pipe is computed by the formula $4v^2 + 5v = 2 + \frac{1200 HD}{L}$. Solve this formula for v .

SUPPLEMENTARY EXERCISES

A second method of solving a quadratic equation by completing the square besides that given in § 147 is shown below.

1. Solve $ax^2 + bx + c = 0$ for x .

SOLUTION. — Multiply through by $4a$, or 4 times the coefficient of x^2 .

Then $4a^2x^2 + 4abx + 4ac = 0$.

Transposing, $4a^2x^2 + 4abx = -4ac$.

Add b^2 , the square of the coefficient of x in the original equation, to both members.

Then $4a^2x^2 + 4abx + b^2 = b^2 - 4ac$.

Taking square root, $2ax + b = \pm \sqrt{b^2 - 4ac}$.

Transposing, $2ax = -b \pm \sqrt{b^2 - 4ac}$.

Dividing by $2a$, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Solve by the method of Problem 1:

2. $3x^2 + 4x + 5 = 0$.

7. $15V^2 - 7 = 9V$.

3. $7n^2 - 3n - 4 = 0$.

8. $6b^2 = b + 5$.

4. $5t^2 - 4t = 9$.

9. $5w^2 - 17w + 6 = 0$.

5. $12K^2 + K = 11$.

10. $3M^2 = 7M + 15$.

6. $9a^2 = 3a + 7$.

11. $6r^2 = 13r - 6$.

CHAPTER XV

SYSTEMS INVOLVING QUADRATIC EQUATIONS

151. Systems involving Quadratic Equations.—Some problems may be expressed and solved by means of a system of two equations containing two unknown numbers in which at least one of the equations is of the second or higher degree. In this chapter we shall discuss only *systems in which one equation is linear and the other quadratic*, these being sufficient for the present needs of the student. A complete treatment of systems involving quadratic and higher equations will be found in the **SECOND COURSE**.

152. Elimination by Substitution.—The easiest method of eliminating one of the unknown numbers in a system consisting of one linear and one quadratic equation is by *substitution*. See § 128.

$$\text{EXAMPLE.}—\text{Solve } \begin{cases} m - 2n = 5, & (1) \\ m^2 + n^2 = 10. & (2) \end{cases}$$

$$\text{Solving (1) for } m, \qquad m = 2n + 5.$$

$$\text{Replacing } m \text{ by } 2n + 5 \text{ in (2), } (2n + 5)^2 + n^2 = 10. \qquad (3)$$

$$\text{Solving (3),} \qquad n = -1 \text{ or } -3.$$

$$\text{Replacing } n \text{ by } -1 \text{ in (1),} \qquad m + 2 = 5,$$

$$m = 3.$$

$$\text{Replacing } n \text{ by } -3 \text{ in (1),} \qquad m + 6 = 5,$$

$$m = -1.$$

Hence there are two solutions :

$$m = 3, \ n = -1, \text{ or } m = -1, \ n = -3.$$

As shown in the above example, elimination of one of the unknown numbers in a system containing a linear and a quadratic equation always leads to a *quadratic equation* in one unknown number, which may be solved by the methods of Chapter XIV.

By substituting each of the values thus found for one of the unknown numbers in the linear equation of the given system, a corresponding value of the other unknown number is found.

Hence, a system consisting of one linear and one quadratic equation has two solutions.

EXERCISES

Solve :

1. $\begin{cases} 2a + b = 4, \\ a^2 + b^2 = 5. \end{cases}$
2. $\begin{cases} 2x + 3y = 0, \\ 3x^2 - y^2 = 23. \end{cases}$
3. $\begin{cases} m + n = 5, \\ mn = 4. \end{cases}$
4. $\begin{cases} A - B = 1, \\ AB = 2. \end{cases}$
5. $\begin{cases} u^2 + uv + v^2 = 39, \\ u - v + 3 = 0. \end{cases}$
6. $\begin{cases} V^2 = 4t, \\ V + 2t = 4. \end{cases}$
7. $\begin{cases} r_1^2 = 4r_2 + 18, \\ 3r_1 = 4r_2 + 24. \end{cases}$
8. $\begin{cases} p^2 + 3pq + q^2 = 22, \\ 2p = q. \end{cases}$
9. $\begin{cases} 2R + r = 14, \\ R^2 + 3Rr = 49. \end{cases}$
10. $\begin{cases} c + 2d = 4, \\ cd + 6 = 0. \end{cases}$
11. $\begin{cases} M + N = 2, \\ M^2 - MN + N^2 = 6. \end{cases}$
12. $\begin{cases} y - 2x = 12, \\ y^2 - x^2 - 3x + 6 = 0. \end{cases}$
13. $\begin{cases} x^2 + y^2 = 74, \\ 3x - 2y = 1. \end{cases}$
14. $\begin{cases} 2W + 2D = 5WD, \\ 2W + 2D = 5. \end{cases}$
15. $\begin{cases} 2h^2 + 3k^2 = 11 + 4hk, \\ h - 5 = 3k. \end{cases}$
16. $\begin{cases} E + F = 7, \\ E^2 - EF + F^2 = 31. \end{cases}$
17. $\begin{cases} a + b = 4, \\ b + a = ab. \end{cases}$
18. $\begin{cases} w - 3 = v, \\ w^2 + 19 + v^2 = 3wv. \end{cases}$
19. $\begin{cases} 8V + 12 = 4t, \\ 3V^2 + 2t^2 = 48 + t. \end{cases}$
20. $\begin{cases} 2x - 5 = x', \\ x + 3x' = 2xx'. \end{cases}$
21. $\begin{cases} (6 - y)(7 + z) = 80, \\ z + y = 5. \end{cases}$
22. $\begin{cases} m = 3n + 1, \\ n^2 + mn = 33. \end{cases}$
23. $\begin{cases} R^2 - RS - 6S^2 = 16, \\ R = 3S + 1. \end{cases}$
24. $\begin{cases} 3T - t - 12 = 0, \\ 9T^2 + t^2 = 72. \end{cases}$

$$25. \begin{cases} \frac{a}{b} + \frac{b}{a} = \frac{13}{6}, \\ a + b = 5. \end{cases}$$

$$26. \begin{cases} M + N = 2, \\ \frac{3}{N} + \frac{2}{M} = 6. \end{cases}$$

$$27. \begin{cases} p^2 + pq + q^2 = 97, \\ 1 - \frac{q}{p} = \frac{19}{p}. \end{cases}$$

$$28. \begin{cases} G = \frac{3}{t}, \\ G + t = 4. \end{cases}$$

$$29. \begin{cases} \frac{v}{u} + \frac{u}{v} = 3\frac{1}{3}, \\ 3v - 2u + 12 = 0. \end{cases}$$

$$30. \begin{cases} 1 - \frac{A}{B} + \frac{A^2}{B^2} = \frac{21}{B^2}, \\ A + B = 9. \end{cases}$$

31. Find two numbers whose difference is 2 and the sum of whose squares is 34.

32. The hypotenuse of a right triangle is 20 feet and the sum of the legs 28 feet. Find the lengths of the legs.

33. A rectangular field is 40 rods longer than it is wide, and its area is 1152 square rods. Find the dimensions of the field.

34. A rectangular field contains 18 acres, and the length of the fence around it is 232 rods. Find the length and width of the field.

35. The base of a triangle is 5 inches longer than the altitude, and the area is 42 square inches. Find the base and altitude.

36. Of two squares the side of one is 7 feet more than the side of the other, and the area of the larger is 161 square feet more than the area of the other. Find the sides of the squares.

37. A bin is to be made to hold 144 cubic feet. It must be 3 feet deep and 3 times as long as wide. Find the dimensions of the bin.

38. Of two machines in a mill one can turn out an order of goods in 3 hours less time than the other, and if both machines are operated at once, they can turn out the goods in 2 hours. How long would it require each machine alone to turn out the goods?

39. If one machine in a mill can produce a quantity of goods in one half of the time required for a machine of smaller capacity to do it, and the two machines together can produce the goods in 4 days, how many days would it take each machine alone to produce the goods?

40. If one boiler would consume a tank of water in 2 hours less time than another, and the two together would exhaust it in 2 hours 55 minutes, how long would the tank of water supply each boiler alone?

41. In any circle whose radius is R , diameter D , and area S ,

$$S = \pi R^2,$$

$$D = 2 R.$$

By eliminating R between the two equations, get the equation between S and D .

42. In any sphere whose radius is R , diameter D , and area S ,

$$S = 4 \pi R^2,$$

$$D = 2 R.$$

By eliminating R between these two equations, derive the equation between S and D .

43. In any right circular cylinder the radius of whose base is R , altitude H , area of cylindrical surface S , and volume V ,

$$S = \pi R H,$$

$$V = \pi R^2 H.$$

Eliminate R between these two equations and get a formula between S , V , and H .

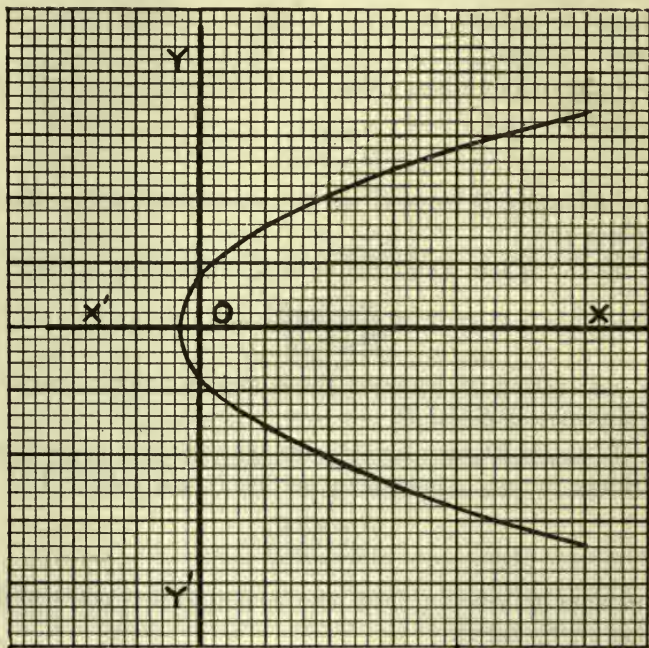
153. Graphs of Quadratic Equations.—It was shown in § 124 that the graph of a linear equation containing two unknown numbers is always a *straight line*. Similarly, it will be found that in general the graph of a quadratic equation containing two unknown numbers is some kind of *curve*.

For example, consider the graph of $y^2 = 9x + 16$.

By assigning values to x , and computing the corresponding approximate values of y , the following sets of values of x and y are obtained:

x	30	20	10	6	2	1	0	-1	$-1\frac{7}{9}$	Greater neg. values
y	± 16.9	± 14	± 10.3	± 8.4	± 5.8	± 5	± 4	± 2.6	0	Imaginary

By locating the points corresponding to these sets of values of x and y and joining them, as in § 124, we get the curve shown below.



Observe that the graph is perfectly symmetrical with reference to the axis XX' . By assigning greater and greater positive values to x , corresponding real values can always be found for y , which shows that the two branches of the curve continue indefinitely to the right. But if negative values are assigned to x larger than $1\frac{7}{8}$, the corresponding values of y are found to be imaginary, which shows that no part of the graph extends farther to the left than $x = -1\frac{7}{8}$.

The above curve is called a *parabola*. It is the kind of path in which astronomers have found that many comets move. Other comets and all of the planets move in closed curves called *ellipses*.

154. Systems solved Graphically.—The solution of a system consisting of one linear and one quadratic equation may be found graphically, as in § 132, by drawing the graphs of both equations

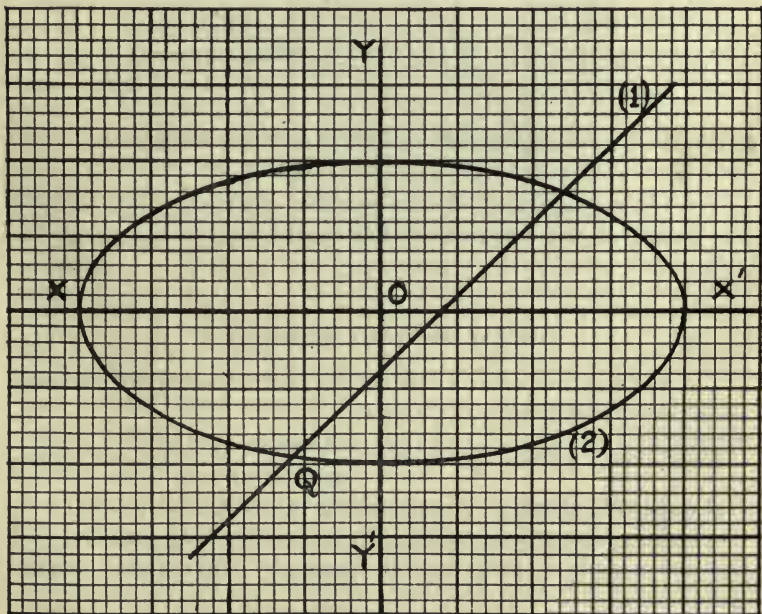
upon the same axes. The sets of values of the unknown numbers corresponding to the points where the graphs meet are the solutions of the system.

EXAMPLE. — Solve graphically $\begin{cases} x - y = 4, & (1) \\ x^2 + 4y^2 = 400. & (2) \end{cases}$

The graph of (1) is the straight line (1) in the figure below. By assigning values to x in (2) the following sets of values of x and y are obtained :

x	0	5	-5	10	-10	15	-15	20	-20
y	± 10	± 9.7	± 9.7	± 8.7	± 8.7	± 6.6	± 6.6	0	0

For either positive or negative values of x greater than 20, y is imaginary. And for either positive or negative values of y greater than 10, x is imaginary. Hence the graph lies entirely within the region bounded by $x = 20$ on the right, $x = -20$ on the left, $y = 10$ above, and $y = -10$ below. The graph of (2) is seen to be the ellipse (2) in the figure.



The straight line (1) meets the curve (2) at two points, P and Q . The set of values $x = 12$, $y = 8$ corresponding to P , and the set of values $x = -5.6$, $y = -9.6$ corresponding to Q , are the two solutions of the system, and will be found to satisfy both equations.

EXERCISES

Draw the graphs of:

1. $x^2 = 4y + 4.$

4. $x^2 - y^2 = 16.$

2. $x^2 + y^2 = 256.$

5. $9x^2 - y^2 = 36.$

3. $4x^2 + 9y^2 = 324.$

6. $xy + 8x - 10y = 80.$

Solve graphically:

7. $\begin{cases} x - y = 2, \\ x^2 + y^2 = 100. \end{cases}$

10. $\begin{cases} 5x^2 + 2y^2 = 532, \\ 3x - y + 10 = 0. \end{cases}$

8. $\begin{cases} y - 2x = 2, \\ 4x^2 + y^2 = 164. \end{cases}$

11. $\begin{cases} 2x - 5y = 3, \\ x^2 - y^2 = 171. \end{cases}$

9. $\begin{cases} x^2 = 2y + 1, \\ x + 2y = 4. \end{cases}$

12. $\begin{cases} 2x - 3y = 10, \\ 3x^2 - 5xy = 2y^2. \end{cases}$

13. If the two graphs of a system did not meet at all, what kind of solutions would the system have?

14. If the graph of the linear equation of a system just touched the graph of the quadratic, what would be the nature of the solutions?

SUPPLEMENTARY EXERCISES

The roots of a quadratic equation in one unknown number may be obtained graphically as follows:

Eliminating y from the system $\begin{cases} x^2 = y, \\ ay + bx + c = 0, \end{cases}$ gives the general quadratic equation in one unknown number $ax^2 + bx + c = 0$. Since the roots of this equation must satisfy both equations of the above system, they may be found by solving this system graphically.

1. Solve $2x^2 + 3x - 20 = 0$.

SOLUTION. — This equation corresponds to the system

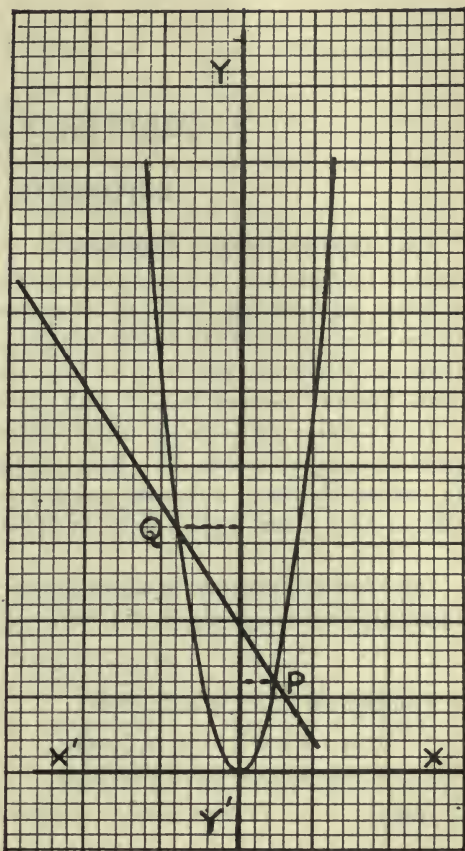
$$\begin{cases} x^2 = y, & (1) \\ 2y + 3x - 20 = 0. & (2) \end{cases}$$

The graph of (1) is the parabola in the figure. The graph of (2) is the straight line meeting the parabola at P and Q . The values of x corresponding to the points P and Q are $2\frac{1}{2}$ and -4 , respectively, which are the roots of the given equation.

NOTE. — Since the equation $x^2 = y$ will be the same for the systems formed from all quadratics, the parabola need be drawn only once and all quadratics solved by use of the one figure. To solve any quadratic in x , substitute y for x^2 , and find two points of the graph of the linear equation formed. Connecting these by a ruler, observe the values of x corresponding to the points where the ruler crosses the parabola. These are the roots of the quadratic.

Draw a large perfect parabola, and by use of it and a ruler find the roots of:

- | | | |
|--------------------------|--------------------------|-------------------------|
| 2. $x^2 - x - 2 = 0$. | 6. $x^2 + x = 12$. | 10. $5x^2 = 1 + 4x$. |
| 3. $x^2 - 2x = 8$. | 7. $x^2 = 2x + 15$. | 11. $6x^2 = x + 35$. |
| 4. $x^2 = 3x + 10$. | 8. $2x^2 - 5x + 2 = 0$. | 12. $4x^2 = 21x + 18$. |
| 5. $x^2 + 7x + 12 = 0$. | 9. $2x^2 + 5x = 3$. | 13. $5x^2 = 3x + 14$. |



CHAPTER XVI

EXPONENTS

155. Positive Integral Exponents. — A positive whole number used to indicate how many times a given number is to be used as a factor was defined in § 5 as an **exponent**. Expressions sometimes are encountered in which negative numbers, zero, and fractions are written in the form of exponents. In this chapter we shall discover the meanings of such expressions.

In the following sections are given some laws which hold for positive integral exponents. Some of these laws have been used in earlier chapters, and are given here primarily for review. Others have been applied only in special cases.

156. Law of Exponents in Multiplication. — The law of exponents in multiplication, as stated in § 48, is expressed in symbols by

$$a^m \times a^n = a^{m+n}.$$

The general proof is as follows:

$$a^m = a \times a \times a \cdots \text{to } m \text{ factors.}$$

$$a^n = a \times a \times a \cdots \text{to } n \text{ factors.}$$

Hence,

$$\begin{aligned} a^m \times a^n &= (a \times a \times a \cdots \text{to } m \text{ factors}) (a \times a \times a \cdots \text{to } n \text{ factors}) \\ &= a \times a \times a \times a \cdots \text{to } m + n \text{ factors} \\ &= a^{m+n}, \text{ by definition of an exponent.} \end{aligned}$$

EXERCISES

Give orally the products of:

1. $a^3 \times a^4$.

3. $x^{12} \times x^8$.

5. $v \times v^9$.

2. $N^6 \times N^5$.

4. $t^{20} \times t^4$.

6. $A^{12} \times A^{12}$.

- | | | |
|-----------------------------------|--|--|
| 7. $r^{10} \times r^{14}$. | 12. $V^4 \times V^7 \times V^9$. | 17. $a^{n+1} \times a^{n-1} \times a^{3n}$. |
| 8. $y^{16} \times y^4$. | 13. $s^{10} \times s^4 \times s^2$. | 18. $P^{2t-3} \times P^{3t+2}$. |
| 9. $P^3 \times P^{12}$. | 14. $Q \times Q^5 \times Q^9$. | 19. $b^{12-a} \times b^{a+3}$. |
| 10. $d^6 \times d^{17}$. | 15. $k^{12} \times k^{12} \times k^{12}$. | 20. $V^{n-m} \times V^{3n+2m}$. |
| 11. $a^2 \times a^5 \times a^3$. | 16. $x^n \times x^{2n} \times x^{3n}$. | 21. $Q^{r+s} \times Q^{r-s}$. |

157. Law of Exponents in Division. — The law of exponents in division was given in § 52. In symbols,

$$a^m \div a^n = a^{m-n}.$$

This follows from § 156, because the quotient a^{m-n} times the divisor a^n equals the dividend a^m .

EXERCISES

Give orally the quotients of:

- | | | |
|---------------------------|----------------------------|---------------------------------|
| 1. $p^6 \div p^2$. | 8. $Q^{32} \div Q^{16}$. | 15. $x^{4n+1} \div x$. |
| 2. $A^{12} \div A^5$. | 9. $v^{22} \div v^{18}$. | 16. $A^{2x+4} \div A^{x+1}$. |
| 3. $w^9 \div w$. | 10. $m^{15} \div m^{11}$. | 17. $y^{4t+10} \div y^8$. |
| 4. $R^{16} \div R^7$. | 11. $B^{28} \div B^{19}$. | 18. $V^{3a+2b} \div V^{2a+b}$. |
| 5. $N^{24} \div N^6$. | 12. $k^{19} \div k^3$. | 19. $Z^m \div Z^{m-2}$. |
| 6. $t^{20} \div t^{12}$. | 13. $W^{5n} \div W^n$. | 20. $G^{t+8} \div G^{t-2}$. |
| 7. $x^{17} \div x^{13}$. | 14. $R^{7a} \div R^{3a}$. | 21. $K^{r+6} \div K^{r-2}$. |

158. Power of a Power. — The law by which a power of some number is itself raised to a power is expressed by,

$$(a^n)^m = a^{mn}.$$

That is, *the m th power of the n th power of any number equals the mn th power of the number.*

For, by definition of an exponent,

$$\begin{aligned} (a^n)^m &= a^n \times a^n \times a^n \cdots \text{to } m \text{ factors} \\ &= a^{n+n+n+\cdots \text{to } m \text{ terms}}, \text{ by § 156} \\ &= a^{mn}. \end{aligned}$$

Thus, $(a^5)^4 = a^{20}$; $(w^6)^3 = w^{18}$; $(p^4)^9 = p^{36}$.

EXERCISES

Give orally :

- | | | | |
|-------------------|----------------------|-----------------------|-------------------------|
| 1. $(a^3)^4$. | 6. $(t^8)^4$. | 11. $(A^{16})^4$. | 16. $(Q^4)^{5t}$. |
| 2. $(x^5)^6$. | 7. $(B^{12})^7$. | 12. $(s^{20})^5$. | 17. $(y^{a+1})^{a-1}$. |
| 3. $(P^4)^3$. | 8. $(k^3)^9$. | 13. $(a^x)^y$. | 18. $(R^{a+b})^{a-b}$. |
| 4. $(r^9)^2$. | 9. $(E^{10})^{10}$. | 14. $(w^n)^{2m}$. | 19. $(U^{2m-n})^n$. |
| 5. $(y^{10})^3$. | 10. $(m^6)^{14}$. | 15. $(x^{2n})^{3m}$. | 20. $(2^2)^4$. |

159. Power of a Product. — The law by which a power of the product of two numbers is found is expressed by,

$$(ab)^n = a^n b^n.$$

That is, *the n th power of the product of two numbers equals the product of the n th powers of the numbers.*

For, by the meaning of an exponent,

$$\begin{aligned} (ab)^n &= (ab)(ab)(ab) \dots \text{to } n \text{ factors} \\ &= (a \times a \times a \dots \text{to } n \text{ factors})(b \times b \times b \dots \text{to } n \text{ factors}) \\ &= a^n b^n. \end{aligned}$$

By similar reasoning the law can be shown to hold for any number of factors.

$$\text{Thus, } (abcd)^5 = a^5 b^5 c^5 d^5; (3xy)^4 = 3^4 x^4 y^4 = 81 x^4 y^4.$$

160. Power of a Fraction. — The law by which a fraction is raised to a power is expressed by,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

That is, *the n th power of a fraction equals the n th power of the numerator divided by the n th power of the denominator.*

$$\begin{aligned} \text{For, } \left(\frac{a}{b}\right)^n &= \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \dots \text{to } n \text{ factors} \\ &= \frac{a \times a \times a \dots \text{to } n \text{ factors}}{b \times b \times b \dots \text{to } n \text{ factors}} \\ &= \frac{a^n}{b^n}. \end{aligned}$$

$$\text{Thus, } \left(\frac{2ab}{3cd}\right)^4 = \frac{(2ab)^4}{(3cd)^4} = \frac{2^4 a^4 b^4}{3^4 c^4 d^4} = \frac{16 a^4 b^4}{81 c^4 d^4}.$$

161. Power of any Monomial. — By use of the laws of exponents in the preceding sections, any power of any monomial may be obtained.

$$\begin{aligned}\text{Thus,} \quad (2 a^3 b^5 c^4)^6 &= 2^6 (a^3)^6 (b^5)^6 (c^4)^6 \text{ by § 159} \\ &= 64 a^{18} b^{30} c^{24} \text{ by § 158.}\end{aligned}$$

$$\begin{aligned}\text{And} \quad \left(\frac{3 m^2 n^7}{5 p^3 q^5}\right)^4 &= \frac{(3 m^2 n^7)^4}{(5 p^3 q^5)^4} \text{ by § 160} \\ &= \frac{81 m^8 n^{28}}{625 p^{12} q^{20}} \text{ by §§ 159 and 158.}\end{aligned}$$

EXERCISES

Give orally:

- | | | | |
|------------------------|---------------------------|--|--|
| 1. $(2 a^6)^3$. | 6. $(w^5 v^2 z^7)^7$. | 11. $\left(\frac{a^4}{b^5}\right)^6$. | 14. $\left(\frac{a^n}{t^{2n}}\right)^m$. |
| 2. $(x^2 y^3)^4$. | 7. $(5 A^2 B^{10})^4$. | 12. $\left(\frac{2 E^6}{P^4}\right)^5$. | 15. $\left(\frac{x^{3a}}{y^{2b}}\right)^4$. |
| 3. $(2 N^3 M^2)^5$. | 8. $(m^5 n^4 p^3)^{10}$. | 13. $\left(\frac{a^2 b^2 c^2}{x^3 y^3 z^3}\right)^8$. | 16. $\left(\frac{N^x M^4}{W^{x+y}}\right)^z$. |
| 4. $(a^2 b^3 c)^6$. | 9. $(3 V^6 P^2)^4$. | | |
| 5. $(P^4 t^2 v^3)^8$. | 10. $(Q^2 n^6)^{12}$. | | |

162. Root of a Power. — To find any root of any power of a base divide the exponent by the index. That is,

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}.$$

$$\text{For, by § 158,} \quad (a^{\frac{m}{n}})^n = a^{\frac{m}{n} \times n} \text{ or } a^m.$$

$$\text{Thus,} \quad \sqrt[5]{x^{20}} = x^4; \quad \sqrt[4]{c^{12}} = c^3; \quad \sqrt[7]{m^{35}} = m^5.$$

EXERCISES

Give orally:

- | | | | |
|-------------------------|---------------------------|---------------------------|---------------------------|
| 1. $\sqrt[4]{a^{24}}$. | 6. $\sqrt[4]{P^{12}}$. | 11. $\sqrt[12]{v^{24}}$. | 16. $\sqrt[7]{s^{42}}$. |
| 2. $\sqrt[5]{y^{15}}$. | 7. $\sqrt[5]{t^{25}}$. | 12. $\sqrt[8]{t^{32}}$. | 17. $\sqrt[8]{W^{56}}$. |
| 3. $\sqrt[8]{N^{24}}$. | 8. $\sqrt[9]{W^{27}}$. | 13. $\sqrt[7]{R^{28}}$. | 18. $\sqrt[3]{k^{36}}$. |
| 4. $\sqrt[6]{x^{18}}$. | 9. $\sqrt[6]{n^{30}}$. | 14. $\sqrt[6]{p^{36}}$. | 19. $\sqrt[12]{G^{48}}$. |
| 5. $\sqrt[3]{z^9}$. | 10. $\sqrt[10]{a^{40}}$. | 15. $\sqrt[9]{B^{27}}$. | 20. $\sqrt[4]{m^{100}}$. |

163. Negative Exponents. — If the law of exponents in division be assumed to hold in case the exponent of the divisor is greater than that of the dividend, its application leads to a new kind of number symbol called a **negative exponent**.

$$\text{Thus,} \quad a^4 \div a^7 = a^{4-7} = a^{-3}; \quad t^6 \div t^{11} = t^{6-11} = t^{-5}.$$

The meaning of a negative exponent is shown as follows:

$$\text{By § 159,} \quad a^3 \div a^8 = a^{3-8} = a^{-5}.$$

$$\text{But by reduction,} \quad a^3 \div a^8 = \frac{a^3}{a^8} = \frac{1}{a^5}.$$

$$\text{Hence,} \quad a^{-5} = \frac{1}{a^5}.$$

$$\text{In general,} \quad a^{-n} = \frac{1}{a^n}.$$

That is, *a negative exponent indicates a power of a base that is to be used as a divisor.*

It is assumed here, and can be proved, that all of the laws that hold for positive integral exponents apply also to negative exponents.

164. Zero Exponent. — If any power of a base is divided by itself by the law in § 159, it leads to a **zero exponent**.

$$\text{Thus,} \quad a^6 \div a^6 = a^{6-6} = a^0; \quad t^8 \div t^8 = t^{8-8} = t^0.$$

$$\text{In general,} \quad a^n \div a^n = a^{n-n} = a^0.$$

$$\text{But} \quad a^n \div a^n = 1.$$

$$\text{Hence,} \quad a^0 = 1.$$

That is, *any base with the exponent 0 equals 1.*

165. Fractional Exponents. — If the law in § 162 be assumed to hold in case the index of the root is not an exact divisor of the exponent, its application leads to a new kind of number symbol called a **fractional exponent**.

$$\text{Thus,} \quad \sqrt[4]{a^3} = a^{\frac{3}{4}}; \quad \sqrt[3]{r^5} = r^{\frac{5}{3}}.$$

$$\text{In general,} \quad a^{\frac{n}{m}} = \sqrt[m]{a^n}.$$

That is, a *fractional exponent indicates a root of a power of a number, the numerator indicating the power, and the denominator indicating the root.*

It may be proved that the laws that apply to positive integral exponents apply also to fractional exponents.

EXERCISES

Simplify :

- | | | |
|--|--|--|
| 1. $n^{-2} \times n^{-3}$. | 14. $P^{\frac{1}{2}} \times P^{\frac{3}{4}}$. | 26. $(m^{-5})^{-6}$. |
| 2. $x^{-5} \times x^{-8}$. | 15. $t^{\frac{1}{3}} \times t^{\frac{1}{4}}$. | 27. $(a^{\frac{1}{2}})^{\frac{1}{2}}$. |
| 3. $w^{12} \times w^{-7}$. | 16. $K^{\frac{5}{9}} \times K^{\frac{3}{4}}$. | 28. $(D^{\frac{3}{4}})^{\frac{1}{2}}$. |
| 4. $p^{-6} \times p^2$. | 17. $n^{\frac{3}{4}} \div n^{\frac{1}{2}}$. | 29. $(k^{\frac{5}{9}})^{\frac{3}{4}}$. |
| 5. $r^{-1} \times r^{-1}$. | 18. $V^{\frac{7}{8}} \div V^{\frac{1}{4}}$. | 30. $(R^{\frac{7}{12}})^{\frac{4}{5}}$. |
| 6. $N^9 \div N^{-8}$. | 19. $R^{\frac{4}{7}} \div R^{\frac{1}{3}}$. | 31. $(V^{\frac{4}{9}})^{\frac{3}{8}}$. |
| 7. $y^{-6} \div y^{-9}$. | 20. $b^{1\frac{1}{2}} \div b^{\frac{2}{3}}$. | 32. $\sqrt{x^{-6}}$. |
| 8. $D^{-5} \div D^3$. | 21. $L^{\frac{2}{3}} \div L^{\frac{1}{4}}$. | 33. $\sqrt[5]{t^{-20}}$. |
| 9. $V^{-12} \div V^{-8}$. | 22. $(x^{-2})^{-3}$. | 34. $\sqrt[6]{V^{-24}}$. |
| 10. $k^4 \div k^{-4}$. | 23. $(a^4)^{-4}$. | 35. $\sqrt[3]{a^{\frac{6}{7}}}$. |
| 11. $T \div T^{-1}$. | 24. $(W^{-1})^5$. | 36. $\sqrt[4]{m^{\frac{8}{9}}}$. |
| 12. $a^{\frac{1}{2}} \times a^{\frac{1}{2}}$. | 25. $(t^7)^{-2}$. | |
| 13. $x^{\frac{2}{3}} \times x^{\frac{1}{3}}$. | | |

SUPPLEMENTARY EXERCISES

Find the value of :

- | | | | | |
|------------------------|-------------------------|-------------------------|---------------|---------------|
| 1. $8^{\frac{2}{3}}$. | 2. $81^{\frac{3}{4}}$. | 3. $64^{\frac{5}{8}}$. | 4. 2^{-1} . | 5. 3^{-2} . |
|------------------------|-------------------------|-------------------------|---------------|---------------|

Simplify :

- | | |
|--|--|
| 6. $n^{\frac{2}{3}} \times n^{-\frac{1}{6}}$. | 11. $(a^{-2} + 3a^{-1} - 4)(a^{-1} - 2)$. |
| 7. $A^{\frac{3}{4}} \div A^{-\frac{1}{2}}$. | 12. $(x^{-1} - 1)(x^{-2} + x^{-1} + 1)$. |
| 8. $(k^{-\frac{5}{12}})^{-\frac{4}{5}}$. | 13. $(t^{-3} + 1) \div (t^{-1} + 1)$. |
| 9. $\sqrt[4]{y^{-\frac{8}{9}}}$. | 14. $(m^{-2} + 2m^{-1}n^{-1} + n^{-2}) \div (m^{-1} + n^{-1})$. |
| 10. $16^{-\frac{3}{4}}$. | 15. $(N^{-3} - 1)^2$. |

MISCELLANEOUS EXERCISES

1. Find the value of W in the formula $W = \frac{2wh}{k+1}$, when $w = 1800$, $h = 26$, and $k = 0.75$.

2. Find the value of A in the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

when $s = 140$, $a = 80$, $b = 90$, and $c = 110$.

3. Simplify the expression $L = \sqrt{2r^2 - r\sqrt{4r^2 - (r\sqrt{2})^2}}$, and thus find a simple formula. Then find L when $r = 12$.

4. Solve for x in the equation $\frac{d}{x+d} = \frac{r-r'}{r}$. Then find the value of x when $d = 24$, $r = 14$, and $r' = 8$.

5. Solve $F = \frac{4\pi^2mr}{t^2}$ for each letter involved.

6. Solve $E = \frac{4n^2L^2w}{g}$ for n , L , and w .

7. In the formulæ $C = \frac{E}{R}$ and $C' = \frac{E}{R+r}$, eliminate R and express E in terms of the remaining letters.

8. Given $P = \frac{nh}{\sqrt{m^2 + n^2}}$, find the value of P when $n = 20$, $h = 28$, and $m = 16$.

9. In $a^2 = b^2 + c^2 - 2bm$, solve for m .

10. In $2x = g - \frac{n-1}{s}x$, solve for x .

11. The volume of a hollow sphere is found by the formula $V = \frac{4}{3}\pi(r^3 - r'^3)$, where r is the radius of the sphere and r' the radius of the hollow. Find the weight of a hollow sphere of brass 8 inches in diameter when the diameter of the hollow is 7 inches. (A cubic inch of brass weighs 0.303 lb.)

12. The area of a ring is found by the formula $A = \pi(r^2 - r'^2)$, where r is the radius of the outer circle that bounds the ring and r' is the radius of the inner circle. When the area of a ring whose outer diameter is 30 inches is 91.1064 square inches, find the inner diameter.

Factor the following expressions:

13. $16a^3 - 12ab + 4ac$.

20. $16x^3y - 2y^4$.

14. $4a^2 + 12ab + 9b^2$.

21. $6m^3 + 12m^2x + 6mx^2$.

15. $9x^2 - 12xy + 4y^2$.

22. $ad + ay + dx + xy$.

16. $25m^4n + 10m^2n^2 + n^3$.

23. $5x^3b - 5x^2b - 30xb$.

17. $64r^2t^2 - 16s^2v^2$.

24. $r^2 - \frac{1}{4}r + \frac{3}{4}$.

18. $16y^3 - 1$.

25. $y^2 + 3\frac{1}{2}yz + \frac{3}{2}z^2$.

19. $m^4n - n^5$.

26. Reduce $\frac{2x^3 - 16x - 6}{3x^3 - 24x - 9}$ to its lowest terms.

27. Reduce $\frac{ax^m - bx^{m+1}}{a^2bx - b^3x^3}$ to its lowest terms.

28. Reduce $\frac{4x^2 - 2x}{2x^2 - x + 1}$ to a mixed expression.

29. Reduce $\frac{m^2(m^4 - n^4) + 3(m^3 - mn^2)}{m^2(m^2 - n^2)}$.

30. Change $1 + \frac{b^2 + c^2 - a^2}{2bc}$ to the form of a fraction.

31. Change $x^2 - 3x - \frac{3x(3-x)}{x-2}$ to a fractional form.

32. Change ab , $\frac{a-b}{a+b}$, and $\frac{a+b}{a-b}$ to fractions having a common denominator.

33. Add $2a$, $3a + \frac{2x}{5}$, and $a - \frac{8x}{9}$.

34. Add $\frac{a+b}{(b-c)(c-a)}$, $\frac{b+c}{(c-a)(a-b)}$, and $\frac{c+a}{(a-b)(b-c)}$.

35. Simplify $\frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)}$.

36. Multiply $\frac{ny^n}{4ax}$ by $\frac{ax}{my^{n-1}}$.

37. Multiply $x^2 - x + 1$ by $\frac{1}{x^2} + \frac{1}{x} + 1$.

38. Find the product of $\frac{1-x^2}{1+y}$, $\frac{1-y^2}{x+x^2}$, and $1 + \frac{x}{1-x}$.

39. Divide $x^4 - \frac{1}{x^4}$ by $x^2 + \frac{1}{x^2}$.

40. Simplify $\left(\frac{c-b}{c+b} - \frac{c^3-b^3}{c^3+b^3}\right) \div \left(\frac{c+b}{c-b} + \frac{c^2+b^2}{c^2-b^2}\right)$.

41. Simplify $\frac{1}{x + \frac{1}{1 + \frac{x+1}{3-x}}}$.

42. Solve $17x - \frac{5x-4}{3} - \frac{8x+4}{5} = 20x - \frac{3x+8}{2} - 5$.

43. Solve $\frac{a-x}{b} - \frac{4a-x}{c} = a-b$ for x .

44. Solve $\frac{1}{ab-ax} + \frac{1}{bc-bx} = \frac{1}{ac-ax}$ for x .

Solve the following systems:

45. $\begin{cases} 3x + 7y = 33, \\ 2x + 4y = 20. \end{cases}$

48. $\begin{cases} 6a + 5b = 112, \\ 8a - 2b = 80. \end{cases}$

46. $\begin{cases} 7x + 3y = 62, \\ 5x - 2y = 36. \end{cases}$

49. $\begin{cases} 6t + 11u = 115, \\ 8t - 22u = -30. \end{cases}$

47. $\begin{cases} 12x + 8y = 116, \\ 2x - y = 3. \end{cases}$

50. $\begin{cases} 2z + 3t = 47, \\ 10z - 12t = -63. \end{cases}$

51. Solve $\begin{cases} \frac{x}{a} - \frac{y}{b} = m, \\ \frac{x}{c} + \frac{y}{d} = n, \end{cases}$ for x and y .

Solve:

$$52. \begin{cases} 8x - 9y - 7z = -36, \\ 12x - y - 3z = 36, \\ 6x - 2y - z = 10. \end{cases} \quad 53. \begin{cases} 7x + 4y - z = 78, \\ 4x - 5y - 3z = -21, \\ x - 3y - 4z = -37. \end{cases}$$

54. When the cost of an article to a merchant is lowered 20%, the merchant who keeps the same selling price makes 30% more. Find his former per cent of gain.

55. By getting a discount of 5% off for cash in buying, a merchant makes a net profit of 8% more. Find his rate of gain when no discount is received.

56. A merchant has two kinds of grain, one at 60 cents per bushel, and the other at 90 cents per bushel, of which he wishes to make a mixture of 40 bushels that may be worth 80 cents per bushel. How many bushels of each must he use?

57. A merchant has three kinds of sugar. He can sell 3 lb. of the first quality, 4 lb. of the second quality, and 2 lb. of the third quality, for 60 cents; or, he can sell 4 lb. of the first quality, 1 lb. of the second quality, and 5 lb. of the third quality, for 59 cents; or, he can sell 1 lb. of the first quality, 10 lb. of the second quality, and 3 lb. of the third quality, for 90 cents. Find the price of each quality.

58. How much cream testing 36% butter fat must be mixed with 20 gallons of milk testing 3% butter fat to have milk testing 4.8% butter fat?

59. Find the square root of $4a^4 - 16a^3 + 24a^2 - 16a + 4$.

60. Find the square root of $m^2 + 2m - 1 - \frac{2}{m} + \frac{1}{m^2}$.

61. Find the square root of 33.1776.

62. Find to three decimal places the square root of 2.

Simplify the following:

63. $\sqrt{96a^2x^3}$.

64. $\sqrt{(x^2 - y^2)(x + y)}$.

65. $\sqrt{\frac{ab^2}{4(a+x)}}$.

66. $\frac{x+1}{x-1}\sqrt{\frac{x-1}{x+1}}$.

67. Multiply $4a\sqrt{\frac{3a^2}{8b}}$ by $\frac{a}{2}\sqrt{\frac{3}{2b}}$.

68. Multiply $a+2\sqrt{b}$ by $a-2\sqrt{b}$.

69. Divide $\sqrt{20} + \sqrt{12}$ by $\sqrt{5} + \sqrt{3}$.

70. Solve $4x^2 + 16x = 33$.

71. Solve $4x - \frac{14-x}{x+1} = 14$.

72. Find to three decimal places the roots of

$$\frac{x}{2} + \frac{2}{x} = \frac{1}{3} + \frac{3}{x}.$$

73. A lady finds that there are just 85 square yards of floor surface to be covered in two of her square bedrooms, one of which is 3 feet longer than the other. How large is each?

74. Solve $\begin{cases} x^2 - y^2 = 23 - 3xy, \\ x + 2y = 7. \end{cases}$ 75. Solve $\begin{cases} 2x^2 - 5y^2 = 20 - xy, \\ 2x - 3y = 1. \end{cases}$

76. A farmer has two small fields, each an exact square. It takes 200 rods of fence to inclose both. Together they contain $8\frac{1}{8}$ acres. Find the dimensions of each.

Simplify and express with positive exponents:

77. $a^{\frac{1}{2}}b^{\frac{3}{2}}c \times a^{\frac{3}{2}}b^{-\frac{1}{2}}c^{\frac{1}{2}}$.

80. $\frac{a^{-2}b^{-3}}{a^{-1}b^2} \times \frac{ab^2}{ab^{-2}}$.

78. $(a^3b^2c)^3 \times (a^{\frac{3}{2}}bc^{\frac{1}{2}})^4$.

81. $\frac{3^n \times 3^{n+1}}{3^{n-1} \times 3^{-2}}$.

79. $(x^{2n}y^m)^2 \div (x^ny^{-m})^2$.

~~X~~ ~~Indur~~
 $x + 3 = \dots$

$x + 3 + x = 85$
 $2x = 82$
 $x = 41$

$41 + 3$

$\begin{array}{r} 44 \\ 41 \\ \hline 85 \end{array}$

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